A GENERAL THEORY OF
RELATIVE REGULAR NEIGHBORHOODS

BY

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Regular neighborhoods of polyhedra in manifolds have been discussed in many papers, most notably, in the present context, in [W], [H-Z], [H] and [S]. We study regular neighborhoods of X mod Y in Z where (Z, X, Y) is an arbitrary triad of finite dimensional, locally compact polyhedra. When applied to manifolds our theory generalizes previous results in that we make no compactness assumptions and place no restrictions on the polyhedra X and Y to be considered. The paper is organized as follows:

1. Definitions and notation
2. Simplicial preliminaries
3. The uniqueness theorem
4. The characterization of collared subpolyhedra
5. Cone-retracting and collaring regular neighborhoods
6. The stellar neighborhood theorem
7. Relationships between regular neighborhoods
8. Collapsibility and collapsible retractions
9. Regular neighborhoods via collapsibility.

V is defined to be a regular neighborhood of X mod Y in Z if there exists a full triangulation $(J, K, L; h)$ of (Z, X, Y) in which V underlies a relative first derived neighborhood, $V = hN(K - L, J')$. Thus, on the one hand, a given regular neighborhood can be presented in an explicit manner with a fixed triangulation. On the other hand, regular neighborhoods are piecewise linear rather than simplicial objects ("there exists a triangulation") and there are uncountably many regular neighborhoods of X mod Y in Z implicit in the choice of triangulating complex and triangulating homeomorphism. We develop the theory by first exploiting the seeming concreteness (§§2–5) and then developing the implicit generality (§§6–9).

After presenting the basic simplicial data we turn immediately to the uniqueness theorem (3.1). (Existence is automatic.) This asserts that, given two regular neighborhoods of X mod Y in Z, there is an ambient piecewise linear isotopy taking one onto the other and keeping $X \cup Y$ fixed. The proof is quite direct and carries a great deal of extra information. For example (3.4) if Z is an n-manifold the isotopy can be realized by a sequence of $2n$ "moves relative to $X \cup Y$."

The boundary of a regular neighborhood has a two-sided collar which "pinches off" near Y (5.3 and 7.5). The material necessary to prove and to utilize this fact is developed in §4. A subsidiary result of this section, irrelevant to the rest of the paper, is the "join-cobordism theorem for polyhedra" (4.3).

The usefulness of a regular neighborhood theory depends on the extent to which it solves the

RECOGNITION PROBLEM. *To be able to recognize when a given polyhedron in Z is a regular neighborhood, or is suitably homeomorphic to a regular neighborhood, of X mod Y in Z.*

For, once V and W are known to be homeomorphic to regular neighborhoods, or to be regular neighborhoods, the uniqueness theorem can be invoked to conclude that they are homeomorphic to each other, or to carry one onto the other by an ambient isotopy.

The heart of this paper is the *stellar neighborhood theorem* (6.1). This says, roughly, that any reasonable candidate for a regular neighborhood (or a regular neighborhood homeomorph) is such if it can be triangulated as a relative stellar neighborhood in its own right—not necessarily as a relative stellar neighborhood in an ambient first derived triangulation, or even as a relative stellar neighborhood in any ambient triangulation. This gives a recognition criterion which is much less stringent than the definition. It makes the theory viable and allows us to answer in §7 many of the practical questions which occur when one wishes to use regular neighborhood theory.

It is useful to notice that, through §7, no mention of collapsing occurs. Thus even when working in manifolds the recognition problem can often be most simply resolved without a collapsing argument.

Finally we turn to collapsibility as a recognition criterion. When the ambient space is a manifold this has been the usual criterion (indeed the definition) because of the correspondence between simplicial collapsing and "shelling" in manifolds. We give a solution to the recognition problem in terms of collapsible retractions (9.1). The existence of a collapsible retraction—i.e. a p.l. retraction with compact, collapsible point inverses—is equivalent (§8) to the existence of a collapse, but it seems to be a more natural notion in the noncompact case. The retraction takes place "all at once" and one does not have to worry about the order in which things collapse.

This paper assumes Whitehead's results .(in (9.2)). Otherwise it is formally independent of and takes a different viewpoint than other developments. However we were strongly motivated by the Hudson-Zeeman paper and by Zeeman's elegant proof [Z] of the uniqueness of compact, absolute first derived neighborhoods.

1. Definitions and notation.

Simplicial conventions. Small Greek letters represent (closed) simplexes. We write $\sigma < J$ if σ is a simplex of the complex J, and $\sigma < (J - K)$ if σ is a simplex of

J but not of K. Lk (σ, J) and St $(\sigma, J) = \sigma$ Lk (σ, J) denote the link and the star of σ in J.

By a *complex* we mean a locally finite simplicial complex in some Euclidean space R^q. The letters J, K, L always represent complexes. The assertion $K < J$ means that K is a subcomplex of J, and the assertion $K \lhd J$ means that K is a *full* subcomplex of J (i.e. $\sigma \cap K$ is a simplex for each $\sigma < J$). We write $K \cong J$ to indicate that K and J are simplicially isomorphic. $|K|$ denotes the space underlying K.

If S is a set (which may or may not be contained in $|J|$) then we denote

$$N(S, J) = \{\sigma < J \mid \sigma \text{ is the face of a simplex of } J \text{ which meets } S\},$$

$$C(S, J) = \{\sigma < J \mid \sigma \cap S = \varnothing\},$$

$$\dot{N}(S, J) = N(S, J) \cap C(S, J).$$

We often write $N(K-L, J_*)$ instead of $N(|K|-|L|, J_*)$, when $K, L < J$ and J_* is a partition of J. In general, we will be quick to write K instead of $|K|$ and $K-L$ instead of $|K|-|L|$, if no confusion can occur.

Two simplexes $\sigma = \sigma^i$ and $\tau = \tau^j$ are *joinable* if their vertices span an $(i+j+1)$-simplex, $\sigma\tau$. Two complexes K and J in R^q are joinable if σ and τ are joinable for each $\sigma < K$, $\tau < J$ and if $(\sigma\tau) \cap (\sigma_1\tau_1) = (\sigma \cap \sigma_1)(\tau \cap \tau_1)$ for each $\sigma, \sigma_1 < K$ and $\tau, \tau_1 < J$. The empty simplex $\varnothing$ belongs to every complex and $\varnothing\sigma = \sigma$ for every simplex σ. The *join* of two joinable complexes is denoted by KJ— or for emphasis by $K * J$—and is given by

$$KJ = K * J = \{\sigma\tau \mid \sigma < K, \tau < J\}.$$

Although KJ is not locally finite in general, we will only deal with locally finite subcomplexes of KJ. If $f: K_1 \to K_2$ and $g: J_1 \to J_2$, where K_i is joinable with J_i, then $(f * g): K_1 J_1 \to K_2 J_2$ is defined by the condition that

$$(f * g)(tx + (1-t)y) = tf(x) + (1-t)g(y)$$

for each $x \in K_1$ and $y \in J_1$.

If $\sigma < J$ then J' denotes a first derived of J and $b(\sigma)$ denotes the distinguished interior point of σ. (It is often important *not* to choose $b(\sigma)$ as the actual barycenter.) If we write $A = b(\sigma_0) \cdots b(\sigma_q) < J'$ it is understood that $\sigma_0 < \sigma_1 < \cdots < \sigma_q$. If $\sigma < J$ then *the dual to σ in J, $D(\sigma, J)$*, and its subcomplex $\dot{D}(\sigma, J)$ are defined by

$$D(\sigma, J) = \{b(\sigma_0) \cdots b(\sigma_q) \mid \sigma < \sigma_0 < \cdots < \sigma_q < J\} < J'$$

and

$$\dot{D}(\sigma, J) = \{b(\sigma_0) \cdots b(\sigma_q) \mid \sigma \lneq \sigma_0 < \cdots < \sigma_q < J\} < J'.$$

We shall repeatedly use the fact that $\dot{D}(\sigma, J) \cong \text{Lk}(\sigma, J)'$ under the isomorphism taking $b(\sigma\alpha_0) b(\sigma\alpha_1) \cdots b(\sigma\alpha_q)$ onto $b(\alpha_0) b(\alpha_1) \cdots b(\alpha_q)$. More generally, if $f: J \to K$ is a simplicial mapping, if $f: J' \to K'$ is also simplicial, and if $\sigma < K$ then

$$D(\sigma, f) \equiv f^{-1} D(\sigma, K)$$

$$= \{b(\tau_0) \cdots b(\tau_q) \mid \sigma < f(\tau_0), \tau_0 < \cdots < \tau_q < J\} < J'.$$

$\dot{D}(\sigma, f)$ is defined correspondingly. These duals are studied in detail in §5 of [C].

Polyhedral conventions. A triangulation of a topological space T is a pair (J, h) where J is a simplicial complex and $h: J \to T$ is a homeomorphism. A *polyhedron* consists of a space X and a maximal family of piecewise-linearly related triangulations of X. When speaking of a *triangulation of a polyhedron* we always mean a triangulation in the given family. If Z and X are polyhedra, with X contained in Z, then X is a *subpolyhedron* of Z ($X < Z$) if there is a triangulation (J, h) of Z and a subcomplex K of J such that $(K, h|K)$ is a triangulation of X. In this case we say that (J, h) is an *extension* of $(K, h|K)$. More generally, if $X < Z$, $Y < Z$, and $\{Z_i\}$ is a family of subpolyhedra of Z then $(J, \{J_i\}, K, L; h)$ is a triangulation of $(Z, \{Z_i\}, X, Y)$ if the J_i, K and L are subcomplexes of J such that $h(J_i) = Z_i$, $h(K) = X$ and $h(L) = Y$. If, further, K and L are full in J then this is called a *full triangulation* of $(Z, \{Z_i\}, X, Y)$.

All mappings and manifolds considered are piecewise linear. We write $X \simeq Y$ to indicate that X and Y are piecewise linearly homeomorphic.

$\mathrm{Bdy}_Z X$ and $\mathrm{Int}_Z X$ denote the topological boundary and interior of X in Z. The notation ∂X is reserved (unless otherwise stipulated) for the boundary of a manifold. The closure of a set S is denoted by $\mathrm{Cl}\,(S)$ or by $\bar{S}$. Finally, if $X, Y < Z$ we denote

$$\mathrm{Cl}\,(X - Y) = X_R; \qquad \mathrm{Cl}\,(X - Y) \cap Y = Y_R.$$

(The "R" stands for "relativized." Hudson-Zeeman denoted these sets by $X_{\natural}$ and $Y_{\natural}$.) This notation will only be used with X's and Y's, so no confusion should occur.

Regular neighborhoods. If $X, Y, V < Z$ then V is a *regular neighborhood of X mod Y in Z* if there exists a full triangulation $(J, K, L; h)$ of (Z, X, Y) and a first derived J' such that $V = hN(K-L, J')$. More generally, $(V, \{V_i\})$ is a regular neighborhood of X mod Y in $(Z, \{Z_i\})$ if there is a full triangulation $(J, \{J_i\}, K, L; h)$ of $(Z, \{Z_i\}, X, Y)$ such that $V = hN(K-L, J')$ and $V_i = hN(K-L, J'_i) \equiv hN(K \cap J_i - L \cap J_i, J'_i)$.

Notice that, if $g: Z \to Z$ is a homeomorphism fixed on $X \cup Y$, and if V is a regular neighborhood of X mod Y in Z via the triangulation $(J, K, L; h)$, as above, then $g(V)$ is a regular neighborhood of X mod Y in Z via the triangulation $(J, K, L; gh)$. Thus, though most authors suppress the triangulating homeomorphism, it will often be essential that we keep track of it.

2. Simplicial preliminaries. In this section we present the basic simplicial data which will be used throughout the paper.

LEMMA 2.1. *If $K \lhd J$ and $Q < J$ then $K \cap Q \lhd Q$.*

LEMMA 2.2. *If $L \lhd K \lhd J$ then every simplex $\sigma < N(K, J)$ is uniquely expressible as $\sigma = \alpha\beta\gamma$ where $\alpha < L$, $\beta < C(L, K)$ and $\gamma < \dot{N}(K, J)$. Moreover if σ is a principal simplex of $N(K, J)$ then $(\alpha\beta) \neq \varnothing$.*

Relative stellar neighborhoods. If $K, L < J$ we denote

$$K_R = N(K-L, K); \qquad L_R = L \cap K_R.$$

This notation will only be used for complexes denoted by K's and L's so no confusion should arise. Since our complexes have the weak topology this notation is connected with the corresponding notation for polyhedra by

LEMMA 2.3. *If $X=|K|$ and $Y=|L|$ then $X_R=|K_R|$ and $Y_R=|L_R|$.*

COROLLARY 2.4. *If $K, L < J$ and J_* is a partition of J inducing partitions K_*, L_* then $(K_*)_R=(K_R)_*$ and $(L_*)_R=(L_R)_*$.*

LEMMA 2.5. *If $K, L < J$ and J_* is a partition of J then*

$$N(K-L, J_*) = \bigcup \{N(K-L, \sigma_*) \mid \sigma < J, \sigma \cap (K-L) \neq \varnothing\}.$$

LEMMA 2.6. *If $K, L < J_0 \triangleleft J$ then $N(K-L, J_0)=N(K-L, J) \cap J_0$.*

Proof. If σ is a principal simplex of $N(K-L, J_0)$ then σ is a simplex of J_0 meeting $K-L$. So $\sigma < N(K-L, J) \cap J_0$. If on the other hand $\sigma < N(K-L, J) \cap J_0$ then $\sigma < J_0$ and there exists a simplex τ such that $\sigma < \tau < J$ and $\tau \cap (K-L) \neq \varnothing$. Then, since J_0 is full, $\tau \cap J_0 = \tau_0$ is a simplex which clearly contains $\sigma \cup (\tau \cap K)$. Thus τ_0 is a simplex of J_0 such that $\sigma < \tau_0$ and $\tau_0 \cap (K-L) \neq \varnothing$. Hence $\sigma < N(K-L, J_0)$, q.e.d.

COROLLARY 2.7. *If $L < J$ and $K \triangleleft J$ then $K_R=K \cap N(K-L, J)$.*

Proof. $K \cap L < K \triangleleft J$. Therefore, by (2.6),

$$K \cap N(K-(L \cap K), J) = N(K-(L \cap K), K) = N(K-L, K) = K_R.$$

COROLLARY 2.8. *If $K, L \triangleleft J$ then*
(1) $L_R \triangleleft K_R \triangleleft N(K-L, J)$,
(2) $N(K-L, J)=N(K_R-L_R, J)=N(K_R, N(K-L, J))$.

Proof. $L \triangleleft J$ and $L_R=L \cap K_R$. Also $K \triangleleft J$ and $K_R=K \cap N(K-L, J)$. Therefore (1) follows from (2.1). Assertion (2) is immediate,   q.e.d.

We come now to the basic fact about relative stellar neighborhoods.

PROPOSITION 2.9. *If $K, L \triangleleft J$ then every simplex σ of $N(K-L, J)$ is uniquely expressible as $\sigma=\alpha\beta\gamma$ where $\alpha < L_R$, $\beta < C(L_R, K_R)=C(L, K)$ and*

$$\gamma < \dot{N}(K_R, N(K-L, J)).$$

Further, if σ is principal then $\beta \neq \varnothing$.

Proof. This is an immediate consequence of (2.8) and (2.2), except for the last assertion. But from (2.2) we see that if σ is principal then $(\alpha\beta) \neq \varnothing$. Since no principal simplex of K_R lies in L_R this implies that $\beta \neq \varnothing$,   q.e.d.

CONVENTION. If $K, L \triangleleft J$ then the assertion that $\alpha\beta\gamma < N(K-L, J)$ carries with it the fact that $\alpha < L_R$, $\beta < C(L_R, K_R)$ and $\gamma < \dot{N}(K_R, N(K-L, J))$.

Proposition 2.9 allows us to make the following

DEFINITION. Let $a=(0, 0)$, $b_0=(1, 0)$ and $b_1=(1, 1) \in R^2$. Let $\Delta=ab_0b_1$. If K and L are full subcomplexes of J then *the natural parameterization of $N(K-L, J)$ is the*

unique simplicial mapping $\eta: N(K-L, J) \to \Delta$ such that $\eta(\alpha)=a$, $\eta(\beta)=b_0$ and $\eta(\gamma)=b_1$, for every simplex $\alpha\beta\gamma < N(K-L, J)$.

Relative first derived neighborhoods. It is clear that $|N(K-L, J')| < |N(K-L, J)|$. This, (2.5) and (2.9) yield

LEMMA 2.10. *If $K, L \vartriangleleft J$ then the simplexes of $N(K-L, J')$ are precisely those simplexes of J' of the form*

$$A = b(\alpha_0\beta_0\gamma_0)b(\alpha_1\beta_1\gamma_1)\cdots b(\alpha_q\beta_q\gamma_q)$$

where $\alpha_i\beta_i\gamma_i < N(K-L, J)$ and, for each i, $\gamma_i = \varnothing$ or $\beta_i \neq \varnothing \neq \gamma_i$. The simplexes of $\dot{N}(K-L, J')$ are precisely those of $N(K-L, J')$ such that $\beta_i = \varnothing = \gamma_i$ or $\beta_i \neq \varnothing \neq \gamma_i$, for each i.

COROLLARY 2.11. *If $K, L \vartriangleleft J$ then $N(K-L, J')$ and $\dot{N}(K-L, J')$ are full in J'. Also, $N(K-L, J') \cap L' = L'_R$.*

LEMMA 2.12. *If $K, L \vartriangleleft J$ then*

$$|\dot{N}(K-L, J')| = (\mathrm{Bdy}_{|J|} |N(K-L, J')|) \cup |L'_R|.$$

LEMMA 2.13. *If $K, L \vartriangleleft J$ and $J_0 < J$ then*

$$N(K-L, J'_0) = \mathrm{Cl}\,[N(K-L, J') \cap J'_0 - L'].$$

Lemmas (2.12) and (2.13) follow from (2.10) and standard arguments about closures of sets in simplicial complexes.

LEMMA 2.14. *Suppose that $f: J_1 \to J_2$ is a simplicial mapping and that $f: J'_1 \to J'_2$ is also simplicial. If K and L are full subcomplexes of J_2 and $f^{-1}(L_R)=(f^{-1}L)_R$ then*

(1) $f^{-1}N(K-L, J'_2)=N(f^{-1}K-f^{-1}L, J'_1)$,

(2) $f^{-1}\dot{N}(K-L, J'_2)=\dot{N}(f^{-1}K-f^{-1}L, J'_1)$.

Proof. Suppose that $A=b(\sigma_0)\cdots b(\sigma_q) < f^{-1}N(K-L, J'_2)$. Then $f(A)=b(f\sigma_0)\cdots b(f\sigma_q)$ is the face of a simplex of J'_2 which meets $K'-L'$. Since $L'_R \vartriangleleft K'_R$ this simplex has a vertex $b(\mu)$ in $K'-L'$. Therefore $b(\mu)f(A) < J'_2$ and there is an integer i such that $f(\sigma_0) < \cdots < f(\sigma_i) < \mu < f(\sigma_{i+1}) < \cdots < f(\sigma_q)$. If $i<q$ there is a simplex $\bar{\mu}$ such that $\sigma_i < \bar{\mu} < \sigma_{i+1}$ and $f(\bar{\mu})=\mu$. Then $b(\bar{\mu})A \cap (f^{-1}K-f^{-1}L) \neq \varnothing$ and we see that $A < N(f^{-1}K-f^{-1}L, J'_1)$. If on the other hand $i=q$ (i.e. $f(\sigma_q)<\mu$) then $f(\sigma_q) < K_R$. Hence

$$\sigma_q < f^{-1}(\mathrm{Cl}\,(K-L)) = f^{-1}((K-L) \cup L_R)$$
$$= (f^{-1}K-f^{-1}L) \cup f^{-1}(L_R)$$
$$= (f^{-1}K-f^{-1}L) \cup (f^{-1}L)_R, \text{ by hypothesis}$$
$$= (f^{-1}K)_R$$

so that, $A < \sigma'_q < N(f^{-1}K-f^{-1}L, J'_1)$.

This proves that $f^{-1}N(K-L, J'_2) < N(f^{-1}K-f^{-1}L, J'_1)$. The opposite inclusion is immediate. This proves (1). Assertion (2) follows from (1) and the fact that $A \cap (f^{-1}K-f^{-1}L) = \varnothing$ if and only if $f(\sigma) \cap (K-L) = \varnothing$, q.e.d.

LEMMA 2.15. *Suppose that $L < K < J$ and that $p: J \to K$ is a simplicial retraction with $p^{-1}L = L$. Then, if $p: J' \to K'$ is also simplicial, and if A is a simplex of L'*

(1) $p^{-1}\dot{N}(L', K') = \dot{N}(L', J')$,

(2) $p^{-1}Lk(A, K') = Lk(A, J')$.

The first assertion is a special case of (2.14). The second assertion is proved by the same type of argument as was used in (2.14).

The basic example.

LEMMA 2.16. *If $\sigma = \alpha\beta\gamma$ is an n-simplex, $\beta \neq \varnothing \neq \gamma$, and if σ_* is a partition of σ then*

(a) $N(\alpha\beta - \alpha, \sigma'_*)$ *is a combinatorial n-ball. (Here $\sigma'_* \equiv (\sigma_*)'$.)*

(b) $N(\alpha\beta - \alpha, \dot{\sigma}'_*)$ *and $\dot{N}(\alpha\beta - \alpha, \sigma'_*)$ are $(n-1)$-balls.*

(c) $\partial N(\alpha\beta - \alpha, \sigma'_*) = N(\alpha\beta - \alpha, \dot{\sigma}'_*) \cup \dot{N}(\alpha\beta - \alpha, \sigma'_*)$.

(d) $N(\alpha\beta - \alpha, \dot{\sigma}'_*) = N(\alpha\beta - \alpha, \sigma'_*) \cap \dot{\sigma}'_*$.

Proof. Assertion (d) is a consequence of (2.6).

Assertions (a)–(c) do not depend on which first derived σ'_* of σ_* we are discussing (i.e., on which points are chosen as barycenters). Thus it will suffice to exhibit a first derived in which $|N(\alpha\beta - \alpha, \sigma'_*)|$ is a convex n-cell, $|\dot{N}(\alpha\beta - \alpha, \sigma'_*)|$ is a convex $(n-1)$-cell in its boundary, and $N(\alpha\beta - \alpha, \dot{\sigma}'_*)$ is the complement of $\dot{N}(\alpha\beta - \alpha, \sigma'_*)$ in $\partial N(\alpha\beta - \alpha, \sigma'_*)$.

Let $\eta: \sigma \to \Delta$ be the natural parametrization of $\sigma = N(\alpha\beta - \alpha, \sigma)$. Choose $\varepsilon > 0$ so small that, if v is a vertex of σ_* not lying in $(\alpha\beta)_*$ and if $\eta(v) = (x_1(v), x_2(v))$, then $\varepsilon < x_2(v)$. Let $b_\varepsilon = (1, \varepsilon)$, let I_ε be the line segment ab_ε and let W_ε be the 2-simplex ab_0b_ε.

Suppose that τ is a simplex of σ_* which does not lie in $(\alpha\beta)_*$. We assert

$$(\uparrow): \eta^{-1}I_\varepsilon \text{ meets Int } \tau \text{ if and only if } \tau \cap (\alpha\beta - \alpha) \neq \varnothing.$$

The proof is by analytic geometry and we leave it to the reader. This assertion allows us to choose a first derived σ'_* of σ_* such that if τ is a simplex of σ_* which does not lie in $(\alpha\beta)_*$ but which does meet $(\alpha\beta - \alpha)$, then $b(\tau) \in \eta^{-1}(I_\varepsilon)$. It now follows, from the linearity of η and the assertion $(\uparrow)$ that $\eta^{-1}W_\varepsilon = |N(\alpha\beta - \alpha, \sigma'_*)|$ and that $\eta^{-1}I_\varepsilon = |\dot{N}(\alpha\beta - \alpha, \sigma'_*)|$. By part (d) of the theorem we know further that $|N(\alpha\beta - \alpha, \dot{\sigma}'_*)| = (\eta^{-1}W_\varepsilon) \cap \dot{\sigma}$.

Now, $\eta^{-1}W_\varepsilon$ is convex and has nonempty interior. Thus it is a combinatorial n-ball, which proves (a). Notice that $\eta^{-1}I_\varepsilon$ separates σ. Also $\eta^{-1}I_\varepsilon$ contains no open sets in σ since η is linear and $I_\varepsilon \cap \eta(\sigma)$ contains no open sets in $\eta(\sigma)$. Therefore $\eta^{-1}I_\varepsilon$ is a convex $(n-1)$-cell dividing σ into two n-balls, one of which is $\eta^{-1}W_\varepsilon$. Thus

$$\partial(\eta^{-1}W_\varepsilon) = (\eta^{-1}I_\varepsilon) \cup ((\eta^{-1}W_\varepsilon) \cap \dot{\sigma}'),$$

$$\partial N(\alpha\beta - \alpha, \sigma'_*) = \dot{N}(\alpha\beta - \alpha, \sigma'_*) \cup N(\alpha\beta - \alpha, \dot{\sigma}'_*).$$

This proves (c). Finally (b) is true because $N(\alpha\beta - \alpha, \dot{\sigma}'_*)$, as the complement of a convex $(n-1)$-cell in the boundary of a convex n-cell, is an $(n-1)$-ball,   q.e.d.

3. **The uniqueness theorem.** If X, $Y < Z$ and $(J, K, L; h)$ is a triangulation of (Z, X, Y) then $hN(K-L, J'')$ is a regular neighborhood of X mod Y in Z. Thus regular neighborhoods always exist. In this section we prove the uniqueness theorem:

THEOREM 3.1. *If V and W are regular neighborhoods of X mod Y in Z then there is an ambient isotopy $G: Z \times I \to Z \times I$ such that*
  (a) $G_0 = 1$,
  (b) $G|(X \cup Y) \times I = 1$,
  (c) $G_1(V) = W$.

ADDENDUM 3.2. *If U is a regular neighborhood of X mod Y in Z such that $(U - Y) \subset (\text{Int}_Z V) \cap (\text{Int}_Z W)$ and if T is any subpolyhedron of Z such that $T \cap (V \cup W) < Y$ then G may be chosen so that also $G|(U \cup T) \times I = 1$.*

The proof of (3.1) is invariant on certain subpolyhedra so that it actually yields:

ADDENDUM 3.3. *If $(V, \{V_j\})$ and $(W, \{W_j\})$ are regular neighborhoods of X mod Y in $(Z, \{Z_j\})$ then there is an ambient isotopy $G: (Z, \{Z_j\}) \times I \to (Z, \{Z_j\}) \times I$ such that*
  (a) $G_0 = 1$,
  (b) $G|(X \cup Y) \times I = 1$,
  (c) $G_1(V) = W$; $G_1(V_j) = W_j$ for all j.

ADDENDUM 3.4. *If $Z - (X \cup Y)$ is an n-manifold then the homeomorphism G_1 can be realized as the composition of $2n$ moves relative to $X \cup Y$.*

Here we have used the

DEFINITION. *If $Z - P$ is an n-manifold and h is a homeomorphism of Z onto itself then h is a move relative to P if there is a family of subpolyhedra $\{Q_i^n\}$ of Z such that:*
  (1) The Q_i^n are n-balls with pairwise disjoint interiors.
  (2) There is a single triangulation of Z in which the Q_i^n are all subcomplexes.
  (3) $(Q_i^n \cap P) < \partial Q_i^n$, for all i.
  (4) $h|(Z - \bigcup_i \text{Int}_Z Q_i^n) \cup P = 1$.
  (5) $h(Q_i^n) = Q_i^n$, for all i.
  (6) If any point of $Q_i^n \cap \partial(Z - P)$ is moved under h then

$$Q_i^n \cap \text{Cl}(\partial(Z - P)) = Q_i^{n-1} \text{ is a face of } Q_i^n \text{ and } (Q_i^{n-1} \cap P) < \partial Q_i^{n-1}.$$

Proof of Theorem 3.1. Let $(J, K, L; h)$ and $(\tilde{J}, \tilde{K}, \tilde{L}; \tilde{h})$ be full triangulations of (Z, X, Y) such that $V = hN(K-L, J')$ and $W = \tilde{h}N(\tilde{K}-\tilde{L}, \tilde{J}')$. Then $\tilde{h}^{-1}h: J \to \tilde{J}$ is a piecewise linear homeomorphism, so there exist partitions J_* and $\tilde{J}_*$ of J and $\tilde{J}$ such that $\tilde{h}^{-1}h: J_* \to \tilde{J}_*$ is a simplicial isomorphism. Let J'_* and $\tilde{J}'_*$ be first barycentric subdivisions. We claim that it suffices to prove the following lemma.

LEMMA. *If $K \triangleleft J$, $L \triangleleft J$ and J_* is a partition of J then there exists an ambient piecewise linear isotopy $H: J \times I \to J \times I$ such that*
  (1) $H_0 = 1$.
  (2) $H_t|(K \cup C(K-L, J)) = 1$ $(0 \leq t \leq 1)$.

(3) $H_t(\sigma) = \sigma$ *for each simplex $\sigma < J$ ($0 \leqq t \leqq 1$).*

(4) $H_1(|N(\sigma \cap K - \sigma \cap L, \sigma'_*)|) = |N(\sigma \cap K - \sigma \cap L, \sigma')|$, *if σ is a simplex of J which does not lie in K but meets $K - L$.*

Sufficiency of the lemma. (2), (4), and (2.5) imply that $H_t|K \cup L = 1$ and $H_1 N(K - L, J'_*) = N(K - L, J')$. Let $\tilde{H}: \tilde{J} \times I \to \tilde{J} \times I$ be the corresponding isotopy of $\tilde{J}$ given by the lemma. Then the desired isotopy $G: Z \times I \to Z \times I$ is defined as the composition of the following homeomorphisms:

$$(Z \times I) \xrightarrow{(h \times 1)^{-1}} (J \times I) \xrightarrow{H^{-1}} (J \times I) \xrightarrow{(h \times I)} (Z \times I)$$

$$\xrightarrow{(\tilde{h} \times 1)^{-1}} (\tilde{J} \times I) \xrightarrow{\tilde{H}} (\tilde{J} \times I) \xrightarrow{(\tilde{h} \times 1)} (Z \times I).$$

This proves the theorem modulo the lemma.

Proof of the lemma. If we can find a homeomorphism H_1 satisfying (2)–(4) then the condition (3), $H_1(\sigma) = \sigma$ for each $\sigma < J$, implies, by Lemma 15 of [Z], that we can find the desired isotopy. Hence our problem is to construct such an H_1.

Let S_i be the set (not a subcomplex) of simplexes $\sigma^j < J$ ($j \leqq i$) which do not lie in K but which meet $K - L$. Thus $S_0 = \varnothing$ and, if $\dim J = n$, then $J = K \cup C(K - L, J) \cup S_n$. Let $J_i = K \cup C(K - L, J) \cup S_i$. Set $f_0 = 1$. Assume inductively that a homeomorphism $f_i: J \to J$ has been defined such that $f_i|K \cup C(K - L, J) = 1$, $f_i(\sigma) = \sigma$ for all $\sigma < J$ and $f_i N(\sigma \cap K - \sigma \cap L, \sigma'_*) = N(\sigma \cap K - \sigma \cap L, \sigma')$ for all $\sigma \in S_i$. We proceed to define f_{i+1} as a composition gf_i for an appropriate homeomorphism g.

Suppose $\sigma = \sigma^{i+1} \in S_{i+1}$. By (2.9), $\sigma = \alpha\beta\gamma$ where $\alpha < L_R$, $\beta < C(L_R, K_R)$ and $\gamma < C(K_R, J)$. By definition of S_{i+1}, neither β nor γ is empty. Notice that

$$(\sigma \cap K) - (\sigma \cap L) = (\sigma \cap K_R) - (\sigma \cap L_R) = \alpha\beta - \alpha.$$

Clearly $\dot{\sigma} < J_i$ and $f_i(\dot{\sigma}) = \dot{\sigma}$. By (2.16) and (2.5)

$$N(\alpha\beta - \alpha, \sigma'_*) \cap \dot{\sigma}'_* = N(\alpha\beta - \alpha, \dot{\sigma}'_*)$$
$$= \bigcup \{N(\tau \cap K - \tau \cap L, \tau'_*) \mid \tau < \dot{\sigma}, \tau \cap (K - L) \neq \varnothing\}$$

and a similar statement holds for $N(\alpha\beta - \alpha, \sigma') \cap \dot{\sigma}'$. Thus, applying f_i to each summand we have

$$f_i(N(\alpha\beta - \alpha, \sigma'_*) \cap \dot{\sigma}'_*) = N(\alpha\beta - \alpha, \sigma') \cap \dot{\sigma}'.$$

Therefore (2.16) $N(\alpha\beta - \alpha, \sigma')$ and $f_i N(\alpha\beta - \alpha, \sigma'_*)$ are $(i+1)$-balls in σ meeting $\dot{\sigma}$ in the common i-ball $N(\alpha\beta - \alpha, \dot{\sigma}')$. Recalling that $N(\alpha\beta - \alpha, \sigma'_*)$ has a convex realization, it follows that $(\sigma, f_i N(\alpha\beta - \alpha, \sigma'_*))$ and $(\sigma, N(\alpha\beta - \alpha, \sigma'))$ are each homeomorphic to the cone on the pair $(\dot{\sigma}, N(\alpha\beta - \alpha, \dot{\sigma}'))$. Therefore there is a homeomorphism $g_\sigma: \sigma \to \sigma$ such that $g_\sigma|\dot{\sigma} = 1$ and $g_\sigma f_i N(\sigma \cap K - \sigma \cap L, \sigma'_*) = N(\sigma \cap K - \sigma \cap L, \sigma')$.

Notice that $N(b(\sigma), J') = b(\sigma)\dot{\sigma}' \dot{D}(\sigma, J)$ where $|b(\sigma)\dot{\sigma}'| = |\sigma|$. We define $g: |J| \to |J|$ by the conditions

(i) $g|N(b(\sigma), J') = g_\sigma * 1_{\dot{D}(\sigma, J)}$ if $\sigma \in (S_{i+1} - S_i)$,

(ii) $g(x) = x$ if $x \notin N(b(\sigma), J')$ for any $\sigma \in S_{i+1} - S_i$.

Then, if $\sigma \in (S_{i+1} - S_i)$, $g|\dot{N}(b(\sigma), J') = (g_\sigma|\dot{\sigma}') * 1_{\dot{D}(\sigma, J)} = 1$. Also,

$$[\text{Int}_{|J|} N(b(\sigma), J')] \cap |J_i| = \varnothing,$$

and if $\sigma \neq \tau$ then $N(b(\sigma), J') \cap N(b(\tau), J') = \dot{N}(b(\sigma), J') \cap \dot{N}(b(\tau), J')$. Therefore g is a well-defined homeomorphism of J onto itself, supported by

$$\bigcup \{N(b(\sigma), J') \mid \sigma \in (S_{i+1} - S_i)\}$$

and fixed on J_i. Finally notice that $g(\tau) = \tau$ for all $\tau < J$. For if $g(x) \neq x$ then $x \in N(b(\sigma), J')$ for some $\sigma \in (S_{i+1} - S_i)$. The carrier of x in J' is of the form AB where $A = b(\sigma_0) \cdots b(\sigma_q) < \dot{D}(\sigma, J)$ and $b(\sigma) < B < b(\sigma)\dot{\sigma}'$. If $A = \varnothing$ then $x \in \text{Int } \sigma$ and $g(x) \in \text{Int } \sigma$. If $A \neq \varnothing$ then $x \in \text{Int } \sigma_q$ and $g(x) \in Ab(\sigma)\dot{\sigma}'$ is also in σ_q since its barycentric coordinate relative to $b(\sigma_q)$ is unchanged. Hence g does not change the carrier of any point in J.

If we define $f_{i+1} = gf_i$ and proceed inductively then the homeomorphism $f_n \equiv H_1$ satisfies conditions (2)–(4),   q.e.d.

Proof of Addendum 3.2. We shall show (7.1 and 7.3) that under the hypothesis of (3.2) V and W are both regular neighborhoods of $U \bmod (Y \cup T)$ in Z. Apply (3.1).

Proof of Addendum 3.3. Condition (3) of the lemma used in the proof—namely that $H_t(\sigma) = \sigma$ for all $\sigma < J, 0 \leqq t \leqq 1$—implies that for any subcomplex $J_i < J$, $H_t(J_i) = J_i$ and $H_1 N(K-L, J'_{i\cdot}) = N(K-L, J'_i)$. Thus the proof of the uniqueness theorem naturally carries within it the proof of Addendum (3.3).

Proof of Addendum 3.4. The homeomorphisms $g_{i+1} = f_{i+1} f_i^{-1}$ constructed in the proof of (3.1) have the properties

(1) $g_n \cdots g_2 g_1 = H_1$,

(2) $g_{i+1} N(b(\sigma), J') = N(b(\sigma), J')$ for each $\sigma \in S_{i+1} - S_i$,

(3) $g_{i+1}|(J - \bigcup \{\text{Int}_{|J|} N(b(\sigma), J')|\sigma \in S_{i+1} - S_i\}) = 1$.

If $\sigma \in (S_{i+1} - S_i)$ then $\sigma \nleqslant (K \cup L)$. So if $J - (K \cup L)$ is an n-manifold, it follows that $\dot{D}(\sigma, J)$ is a ball or a sphere, depending on whether or not $\sigma < \text{Cl}\,[\partial(J-(K \cup L))]$ $\equiv J_0$. Then $Q^n(\sigma) \equiv N(b(\sigma), J') = b(\sigma)\dot{\sigma}'\dot{D}(\sigma, J)$ is an n-ball meeting $(K \cup L)$ only in its boundary. If any point of $Q^n(\sigma) \cap J_0$ is moved by g_{i+1} then $\sigma < J_0$ and $Q^n(\sigma) \cap J_0 = N(b(\sigma), J'_0)$ is an $(n-1)$-ball. One checks easily now that each g_{i+1} is a move. Thus H_1 is a composition of n moves, $\tilde{H}_1$ is a composition of n moves and G_1 is a composition of $2n$ moves.

4. The characterization of collared subpolyhedra.

DEFINITION. If $X < Z$ then we say that X is *collared in* Z (X *is bicollared in* Z) if there is a polyhedral neighborhood V of X in Z and a homeomorphism $h: X \times [0, 1]$ $\rightarrow V$ ($h: X \times [-1, 1] \rightarrow V$) such that $h(x, 0) = x$ for all $x \in X$. We say that X *is locally collared in* Z (X *is locally bicollared in* Z) if, for each $x \in X$ there is a polyhedral neighborhood V of x in Z such that $V \cap X$ is collared (bicollared) in V.

In this section we characterize (4.2) the collaring of $X - Y$ in $Z - Y$ when $X, Y < Z$. This result will be used in §5 and §6.

LEMMA 4.1. *Assume that J_1 and J_2 are complexes (in some Euclidean space) and $\sigma < L < (J_1 \cap J_2)$. If there is a homeomorphism $h: J_1 \to J_2$ such that $h|L = 1$ then there is a homeomorphism $g: \mathrm{Lk}\,(\sigma, J_1) \to \mathrm{Lk}\,(\sigma, J_2)$ such that $g|\mathrm{Lk}\,(\sigma, L) = 1$.*

Proof. Suppose that $\sigma = v$ is a vertex of L. Let J_{1*} and J_{2*} be partitions of J_1 and J_2 such that $h: J_{1*} \to J_{2*}$ is a simplicial isomorphism. Then there are homeomorphisms $g_i: \mathrm{Lk}\,(v, J_{i*}) \to \mathrm{Lk}\,(v, J_i)$, $(i = 1, 2)$ defined by pseudo-radial projection (e.g., see Lemma 8 of [Z]) such that

(1) if $v < \alpha < J_i$ then $g_i(|\mathrm{Lk}\,(v, \alpha_*)|) = |\mathrm{Lk}\,(v, \alpha)|$,

(2) $g_1|\mathrm{Lk}\,(v, L_*) = g_2|\mathrm{Lk}\,(v, L_*)$.

So $g = g_2 h g_1^{-1}: \mathrm{Lk}\,(v, J_1) \to \mathrm{Lk}\,(v, J_2)$ is a homeomorphism with $g|\mathrm{Lk}\,(v, L) = 1$.

Suppose that $n > 0$ and the lemma is true for all simplexes of dimension less than n. Let $\sigma = \sigma^n = v\tau < L$. Let $\bar{J}_i = \mathrm{Lk}\,(\tau, J_i)$ and $\bar{L} = \mathrm{Lk}\,(\tau, L)$. By induction hypothesis there is a homeomorphism $\bar{g}: \bar{J}_1 \to \bar{J}_2$ such that $\bar{g}|\bar{L} = 1$. Now v is a vertex of $\bar{L}$, so by the case $n = 0$, there is a homeomorphism $g: \mathrm{Lk}\,(v, \bar{J}_1) \to \mathrm{Lk}\,(v, \bar{J}_2)$ such that $g|\mathrm{Lk}\,(v, \bar{L}) = 1$. Since $\mathrm{Lk}\,(v, \mathrm{Lk}\,(\tau, T)) = \mathrm{Lk}\,(v\tau, T)$, this completes the proof.

PROPOSITION 4.2. *If X and Y are subpolyhedra of Z, the following statements are equivalent.*

(A) *$X - Y$ is collared in $Z - Y$.*

(B) *$X - Y$ is locally collared in Z.*

(C) *If $(J, K, L; h)$ is any triangulation of (Z, X, Y) and if σ is any simplex of $K - L$ then there is a homeomorphism $g: \mathrm{Lk}\,(\sigma, J) \to v\,\mathrm{Lk}\,(\sigma, K)$ such that $g|\mathrm{Lk}\,(\sigma, K) = 1$.*

(D) *If $(J, K, L; h)$ is a full triangulation of (Z, X, Y) then there is a homeomorphism*

$$g: C(L', N(K - L, J')) \to C(L', K') \times I$$

such that

(D.1) *$g(x) = (x, 0)$ for all $x \in C(L', K')$,*

(D.2) *$gD(\sigma, J) = D(\sigma, K) \times I$ for all $\sigma < K - L$,*

(D.3) *$g(C_\sigma) = D(\sigma, K) \times 1$, for all $\sigma < K - L$ where $C_\sigma \equiv \dot{N}(D(\sigma, K), D(\sigma, J))$ $= C(\dot{D}(\sigma, K), \dot{D}(\sigma, J))$([2]).*

Proof. (A) $\Rightarrow$ (B). This is obvious.

(B) $\Rightarrow$ (C). Assume first that $\sigma = x$ is a vertex of $K - L$. If K_* is any partition of K then $\mathrm{St}\,(x, K_*) \times I = P$ can be triangulated as

$$(x, 0) * [(\mathrm{Lk}\,(x, K_*) \times I) \cup (\mathrm{St}\,(x, K_*) \times 1)].$$

Identify $K_* = K_* \times 0$. Obviously there is a homeomorphism

$$g: (\mathrm{Lk}\,(x, P), \mathrm{Lk}\,(x, K_*)) \to (v\,\mathrm{Lk}\,(x, K_*), \mathrm{Lk}\,(x, K_*))$$

([2]) Using (7.5) one can further prove that $N(K - L, J')$ is (p.l.) homeomorphic to a regular neighborhood of $X_R \times 0 \bmod Y_R \times 0$ in $X_R \times I$ and is topologically equivalent to $(X_R \times I)/[(y, 0) = (y, t)$ if $y \in Y_R$ and $t \in I]$.

such that $g|\text{Lk}\,(x, K_*) = 1$. Since $K-L$ is locally collared in J at x and since the pair $(\text{Lk}\,(x, J), \text{Lk}\,(x, K)) \simeq (\text{Lk}\,(x, J_*), \text{Lk}\,(x, K_*))$ for any partition (J_*, K_*) of (J, K) it follows easily that there is a homeomorphism $g: \text{Lk}\,(x, J) \to v\,\text{Lk}\,(x, K)$ such that $g|\text{Lk}\,(x, K) = 1$.

If σ is an arbitrary simplex of $K-L$ we star J at $b(\sigma)$ and apply the above result to $b(\sigma)$. Thus there is a homeomorphism $g: \dot\sigma\,\text{Lk}\,(\sigma, J) \to v\dot\sigma\,\text{Lk}\,(\sigma, K)$ such that $g|\dot\sigma\,\text{Lk}\,(\sigma, K) = 1$. We extend this homeomorphism conewise to a homeomorphism $g_1: \sigma\,\text{Lk}\,(\sigma, J) = v\sigma\,\text{Lk}\,(\sigma, K)$ such that $g_1|\sigma\,\text{Lk}\,(\sigma, K) = 1$. Finally we apply (4.1) and get the desired homeomorphism $g_2: \text{Lk}\,(\sigma, J) \to v\,\text{Lk}\,(\sigma, K)$.

(C) $\Rightarrow$ (D). We define the homeomorphism g inductively. Let

$$J_i = \bigcup \{D(\sigma^{n-i}, J) \mid \sigma^{n-i} < K-L\} < J',$$
$$K_i = \bigcup \{D(\sigma^{n-i}, K) \mid \sigma^{n-i} < K-L\} < K',$$

where n is the dimension of $K-L$. Notice (see (2.10)) that $J_n = C(L', N(K-L, J'))$ and $K_n = C(L', K')$. Suppose that a homeomorphism $g_{i-1}: J_{i-1} \to K_{i-1} \times I$ has been constructed to satisfy (D.1)–(D.3) for all simplexes of dimension greater than $(n-i)$. Let $\sigma = \sigma^{n-i} < K-L$. Notice that

$$D(\sigma, J) \cap J_{i-1} = \bigcup \{D(\tau, J) \mid \sigma \lneqq \tau < K-L\} = N(\dot D(\sigma, K), \dot D(\sigma, J)),$$
$$\dot D(\sigma, K) = \bigcup \{D(\tau, K) \mid \sigma \lneqq \tau < K-L\}$$

and

$$\dot N(\dot D(\sigma, K), \dot D(\sigma, J)) = \bigcup \{C_\tau \mid \sigma \lneqq \tau < K-L\}.$$

Hence, using (D.1)–(D.3), we have

$$g_{i-1}(D(\sigma, J) \cap \dot J_{i-1}) = \dot D(\sigma, K) \times I, \qquad g_{i-1}|\dot D(\sigma, K) = 1$$

and

$$g_{i-1}\dot N(\dot D(\sigma, K), \dot D(\sigma, J)) = \dot D(\sigma, K) \times 1.$$

We first extend g_{i-1} over C_σ. The natural isomorphism of $(\dot D(\sigma, J), \dot D(\sigma, K))$ onto $(\text{Lk}\,(\sigma, J)', \text{Lk}\,(\sigma, K)')$ takes $N(\dot D(\sigma, K), \dot D(\sigma, J))$ onto the regular neighborhood $N(\text{Lk}\,(\sigma, K)', \text{Lk}\,(\sigma, J)')$ of $\text{Lk}\,(\sigma, K)$ in $\text{Lk}\,(\sigma, J)$. But by (C), $\text{Lk}\,(\sigma, J) \simeq v\,\text{Lk}\,(\sigma, K)$ and we know that the complement of a regular neighborhood of T in vT is just a smaller concentric cone. Thus C_σ is piecewise linearly a cone on its subcomplex $\dot N(\dot D(\sigma, K), \dot D(\sigma, J))$. On the other hand, $D(\sigma, K) \times 1 = (b(\sigma), 1) * (\dot D(\sigma, K) \times 1)$. Thus we may extend g_{i-1} conewise over C_σ and obtain a homeomorphism of $\dot D(\sigma, J)$ onto $(\dot D(\sigma, K) \times I) \cup (D(\sigma, K) \times 1)$. Finally we notice that $D(\sigma, J) = b(\sigma)\dot D(\sigma, J)$ and the cell complex $D(\sigma, K) \times I$ can be triangulated as $(b(\sigma), 0) * [(\dot D(\sigma, K) \times I) \cup (D(\sigma, K) \times 1)]$. Thus we may extend g_{i-1} conewise to all of $D(\sigma, J)$. Doing this simultaneously for all σ^{n-i} we construct a homeomorphism g_i satisfying (D.1)–(D.3). Finally we set $g = g_n$.

(D) $\Rightarrow$ (B). Suppose $x \in X - Y$. Choose a full triangulation $(J, K, L; h)$ of (Z, X, Y) in which $v = h^{-1}(x)$ is a vertex. Then by (D), $(D(v, J), D(v, K)) \simeq (D(v, K) \times I, D(v, K) \times 0)$. Hence X is locally collared in Z at $x = h(v)$.

(D) $\Rightarrow$ (A). Let $\tilde{Z} = Z - Y$, $\tilde{X} = X - Y$ and $\tilde{Y} = \varnothing$. Let $(\tilde{A})$–$(\tilde{D})$ be the assertions analogous to (A)–(D). Then (D) $\Rightarrow$ (B) $\Rightarrow$ ($\tilde{B}$) $\Rightarrow$ ($\tilde{D}$). However ($\tilde{D}$) tells us that $C(\tilde{L}, N(\tilde{K} - \tilde{L}, \tilde{J}')) = N(\tilde{K}', \tilde{J}')$ is a collar of $\tilde{K}$. Thus $\tilde{X} = X - Y$ is collared in $\tilde{Z} = Z - Y$,   q.e.d.

As a consequence of (4.2) we get the following "join-cobordism theorem for polyhedra." This is a generalization of Theorem (10.3) of [C].

COROLLARY 4.3. *Suppose that K and L are disjoint, nonempty subcomplexes of J, that each of these complexes is locally collared in J and that every simplex in J is the join of a simplex in K and a simplex in L. Then there is a homeomorphism $h: (J, K, L) \to (K \times I, K \times 0, K \times 1)$.*

Proof. By (4.2D) there is a homeomorphism $f: N(K', J') \to K' \times I$ such that $f(K') = K' \times 0$ and $f\dot{N}(K', J') = K' \times 1$. Similarly there is a homeomorphism g of $N(L', J')$ onto $L' \times I$ such that $g(L') = L' \times 1$ and $g\dot{N}(L', J') = L' \times 0$. Define $h: J \to K \times [0, 2]$ by stipulating that $h|N(K', J') = f$ and $h(g^{-1}(x, t)) = (hg^{-1}(x, 0)) + (0, t)$, where $K < R^q$ and $(0, t) \in R^q \times I$. This is a well-defined homeomorphism because $\dot{N}(K', J') = \dot{N}(L', J')$,   q.e.d.

5. Cone-retracting and collaring regular neighborhoods.

THEOREM 5.1. *If V is a regular neighborhood of X mod Y in Z then there is a retraction $p: V \to X_R$ such that*

(a) *If $x \in X - Y$ then $p^{-1}(x)$ is a compact cone with x as cone point and $p^{-1}(x) \cap \text{Bdy}_Z V$ as base.*

(b) $p^{-1}(Y_R) = Y_R$.

ADDENDUM 5.2. *If $X - Y$ is a proper locally unknotted n-manifold in the q-manifold $Z - Y$ then the projection may be chosen so that, for each $x \in X - Y$, the base of the cone $p^{-1}(x)$ is homeomorphic to $S^{q-n-1} \times B^i$ where $i = i(x)$ is an integer depending on x.*

Proof of (5.1). It suffices (see (2.8) and (2.12)) to prove that, if $L \triangleleft K \triangleleft J = N(K-L, J)$, then there is a retraction p of $N(K-L, J')$ onto K' such that $p^{-1}(x) = x$ if $x \in L$ and $p^{-1}(x)$ is a compact p.l. cone on $p^{-1}(x) \cap \dot{N}(K-L, J')$ otherwise.

Let $\dot{N} = \dot{N}(K, J)$. We may assume without loss of generality that $K < (R^n \times 0)$ and $v\dot{N} < (0 \times R^p)$ are complexes in $R^{n+p} = R^n \times R^p$, where $v = (0, 0)$ is thought of as the origin in R^p. Consider the polyhedron $Q < R^{n+p}$ given by

$$Q = \bigcup \{\alpha * (\beta \times v\gamma) \mid \alpha\beta\gamma < J = N(K-L, J)\}.$$

Here $\alpha * (\beta \times v\gamma)$ is the convex hull of the simplex α and the convex cell $(\beta \times v\gamma)$. It is the union of all line segments from points of α to points of $(\beta \times v\gamma)$. (Indeed if S is a complex partitioning $(\beta \times v\gamma)$ then α and S are joinable.) If $\alpha = \varnothing$ then $\alpha * (\beta \times v\gamma)$

$\equiv (\beta \times v\gamma)$. Let $\bar{q}: R^n \times R^p \to R^n$ be the natural projection. Let $q = \bar{q}|Q$. Then, if $\alpha\beta\gamma < J$,

$$q(\alpha * (\beta \times v\gamma)) = q\{t(a, 0)+(1-t)(b, c) \mid (a, b, c, t) \in (\alpha \times \beta \times v\gamma \times I)\}$$
$$= \{t(a, 0)+(1-t)(b, 0) \mid (a, b, t) \in (\alpha \times \beta \times I)\}$$
$$= \alpha * \beta.$$

Thus q is a retraction of Q onto K. Clearly $q^{-1}L = L$. If x is a point of $K - L$ with carrier $\alpha_0\beta_0$, say $x = ta + (1-t)b$, then

$$q^{-1}(x) = \{t(a, 0)+(1-t)(b, c) \mid c \in v\gamma, \alpha_0\beta_0\gamma < J\}$$
$$= t(a, 0)+(1-t)[b \times v * (\text{Lk}(\alpha_0\beta_0, J) \cap \dot{N})].$$

Therefore $(q^{-1}(x), q^{-1}(x) \cap \bigcup \{\alpha * (\beta \times \gamma) \mid \alpha\beta\gamma < J\}) \simeq (vP, P)$ where

$$P = \text{Lk}(\alpha_0\beta_0, J) \cap \dot{N}(K, J).$$

Let us define a *cell complex* to consist of a polyhedron T together with a covering family of subpolyhedra—each a piecewise linear ball—such that the interiors of these balls are pairwise disjoint, the family is closed under finite intersection, and the boundary of each of these balls is the union of balls in the family. If two cell complexes are isomorphic in the sense that there is an incidence preserving one-one correspondence between the covering families then one can easily build a piecewise linear homeomorphism between the underlying polyhedra which carries each cell onto its corresponding cell.

Now Q is a cell complex with typical cells of the form $\alpha * (\beta \times v\gamma)$ or $\alpha * (\beta \times \gamma)$ or $\alpha\beta$, where $\alpha\beta\gamma$ is an arbitrary simplex of $N(K-L, J)$. Similarly (see (2.16)) $N(K-L, J')$ is a cell-complex with typical cells of the form $N(\alpha\beta - \alpha, (\alpha\beta\gamma)')$ or $\dot{N}(\alpha\beta - \alpha, (\alpha\beta\gamma)')$ or $(\alpha\beta)$ where $\alpha\beta\gamma < N(K-L, J)$. The obvious isomorphism between these complexes induces a homeomorphism $h: (N(K-L, J'), K) \to (Q, K)$ which is the identity on K and takes $(\dot{N}(K-L, J'))$ onto

$$\bigcup \{\alpha * (\beta \times \gamma) \mid \alpha\beta\gamma < N(K-L, J)\}.$$

Then $p = h^{-1}qh$ has the desired properties,   q.e.d.

Proof of (5.2). Suppose that $(J, K, L; h)$ is a full triangulation of (Z, X, Y) with $V = hN(K-L, J')$. Let $x \in (X - Y)$ and let $\alpha\beta$ be the carrier of $h^{-1}(x)$ in J. The proof of (5.1) shows that the projection $p: V \to X_R$, determined by this triangulation, has the property that $p^{-1}(x)$ is the cone on P where

$$P \simeq \text{Lk}(\alpha\beta, N(K-L, J)) \cap \dot{N}(K_R, N(K-L, J)).$$

Assume now that we had chosen $(J, K, L; h) = (J'_0, K'_0, L'_0; h)$ where $(J_0, K_0, L_0; h)$ was already a full triangulation of (Z, X, Y). Thus $\alpha\beta = b(A_0B_0)\cdots b(A_rB_r)$ where $A_i < (L_0)_R$, $B_i < C(L_0, K_0)$ and $B_r \neq \varnothing$. So

$$P \simeq \{b(A_rB_rD_0)\cdots b(A_rB_rD_q) \mid D_i < (\text{Lk}(A_rB_r, J_0) - \text{Lk}(A_rB_r, K_0))\}$$
$$\simeq C(\text{Lk}(A_rB_r, K)', \text{Lk}(A_rB_r, J)').$$

Since $(Z - Y, X - Y)$ is a proper locally unknotted manifold pair, this is homeomorphic to the complement of the regular neighborhood of B^{n-j-1} in B^{q-j-1}, or of S^{n-j-1} in S^{q-j-1}, where j is the dimension of $A_r B_r$,   q.e.d.

THEOREM 5.3. *If V is a regular neighborhood of X mod Y in Z then $(\mathrm{Bdy}_Z V) - Y$ is bicollared in $Z - (X \cup Y)$.*

COROLLARY 5.4. *If $(Z - Y)$ is an n-manifold and V is a regular neighborhood of X mod Y in Z then $(V - Y)$ and $(Z - V) - Y$ are n-manifolds. The second conclusion holds under the weaker assumption that $Z - (X \cup Y)$ is an n-manifold.*

REMARK.   When Z is a manifold there are certain important pairs of polyhedra X, Y—e.g. the link-collapsible pairs studied in [H-Z]—for which V itself is a manifold.

Proof of (5.3). Let $(J, K, L; h)$ be a full triangulation of (Z, X, Y) such that $V = hN(K - L, J')$. We shall use (4.2c) to show that $\dot{N}(K - L, J') - L'$ is collared in $C(K - L, J') - L'$ and in $N(K - L, J') - (K' \cup L')$. This is sufficient by (2.12).

Let A be a simplex of $\dot{N}(K - L, J') - L'$. Then $A = b(\sigma_0) \cdots b(\sigma_q)$ where $\sigma_i = \alpha_i \beta_i \gamma_i < N(K - L, J)$. Moreover $\beta_q \neq \varnothing \neq \gamma_q$, and β_i is empty if and only if γ_i is empty. Hence $A = b(\alpha_0) \cdots b(\alpha_{r-1}) b(\sigma_r) \cdots b(\sigma_q)$ where $\beta_r \neq \varnothing \neq \gamma_r$. Decomposing $\mathrm{Lk}\,(A, J')$ into the blocks determined by its vertices we have

$$\mathrm{Lk}\,(A, J') = \mathrm{Lk}\,(b(\alpha_0) \cdots b(\alpha_{r-1}) b(\sigma_r), \sigma_r') * \dot{D}(\sigma_r, \dot{\sigma}_{r+1}) \cdots \dot{D}(\sigma_{q-1}, \dot{\sigma}_q) * \dot{D}(\sigma_q, J)$$
$$= \mathrm{Lk}\,(B, \sigma_r') * S$$

where $B = b(\alpha_0) \cdots b(\alpha_{r-1}) b(\sigma_r)$ and $S = \dot{D}(\sigma_r, \dot{\sigma}_{r+1}) \cdots \dot{D}(\sigma_q, J)$.

We assert that

(1) $\mathrm{Lk}\,(A, \dot{N}(K - L, J')) = \mathrm{Lk}\,(B, \dot{N}(\alpha_r \beta_r - \alpha_r, \sigma_r')) * S$,

(2) $\mathrm{Lk}\,(A, N(K - L, J')) = \mathrm{Lk}\,(B, N(\alpha_r \beta_r - \alpha_r, \sigma_r')) * S$,

(3) $\mathrm{Lk}\,(A, C(K - L, J')) = \mathrm{Lk}\,(B, C(\alpha_r \beta_r - \alpha_r, \sigma_r')) * S$.

Let $N_r = N(\alpha_r \beta_r - \alpha_r, \sigma_r')$ and $C_r = C(\alpha_r \beta_r - \alpha_r, \sigma_r')$. Consider the first assertion. Every simplex in S lies in $\dot{N}(K - L, J')$ because, for each vertex $b(\sigma_i) = b(\alpha_i \beta_i \gamma_i)$ with $i > r$, $\beta_i \neq \varnothing \neq \gamma_i$. Also $\dot{N}_r < \dot{N}(K - L, J')$. Hence, the right-hand side of (1) is contained in $\dot{N}(K - L, J') \cap \mathrm{Lk}\,(A, J')$. This intersection is equal to $\mathrm{Lk}\,(A, \dot{N}(K - L, J'))$ because $\dot{N}(K - L, J')$ is full in J'. Therefore the right-hand term is contained in the left-hand term. On the other hand, a simplex in this intersection is of the form $D_1 D_2$ where $D_1 < \mathrm{Lk}\,(B, \sigma_r')$, $D_2 < S$ and $D_1 < [\dot{N}(K - L, J') \cap \sigma_r'] = \dot{N}_r$. Therefore $D_1 < \mathrm{Lk}\,(B, \sigma_r') \cap \dot{N}_r = \mathrm{Lk}\,(B, \dot{N}_r)$. Thus $D_1 D_2$ is a simplex of the right-hand side. This proves (1). Assertions (2) and (3) are proved similarly.

We wish to show that $\mathrm{Lk}\,(A, N(K - L, J'))$ is piecewise linearly a cone on $\mathrm{Lk}\,(A, \dot{N}(K - L, J'))$. Using (1) and (2) it will suffice to show that $\mathrm{Lk}\,(B, N_r)$ is a cone on $\mathrm{Lk}\,(B, \dot{N}_r)$. Now N_r is a ball and $\dot{N}_r$ is a face of this ball by (2.16). Since the vertex $b(\sigma_r)$ of B lies in the interior of $\dot{N}_r$ we see that $\mathrm{Lk}\,(B, \dot{N}_r) = \mathrm{Lk}\,(B, \partial N_r) = \partial \mathrm{Lk}\,(B, N_r)$. Therefore $\mathrm{Lk}\,(B, N_r)$ is homeomorphic to $v\,\mathrm{Lk}\,(B, \dot{N}_r)$ by a

homeomorphism which is the identity on Lk $(B, \dot{N}_r)$. This completes the proof that $(\mathrm{Bdy}_Z V) - Y$ is collared in $V - (X \cup Y)$.

One proves similarly that $(\mathrm{Bdy}_Z V) - Y$ is collared in $\mathrm{Cl}((Z - V)) - Y$, q.e.d.

Proof of (5.4). If, in the preceding proof, $J - L$ is an n-manifold then $\dot{D}(\sigma_q, J) \cong \mathrm{Lk}(\sigma_q, J)'$ is a ball or a sphere, for each $\sigma_q < J - L$. Thus formulas (1)–(3) imply that $(N(K - L, J') - L')$ and $(C(K - L, J') - L')$ are n-manifolds, q.e.d.

EXAMPLE 5.5. One might hope that regular neighborhoods could be characterized by Theorems (5.1) and (5.3). This is not possible, as the following example shows. (The matter is further discussed in §9.)

Let $D^3 = [-1, 1]^3 < R^3$ and let A be a polygonal arc which consists of the straight line from $(-1, 0, 0)$ to $(\frac{1}{2}, 0, 0)$ plus a knotted arc from $(\frac{1}{2}, 0, 0)$ to $(\frac{3}{4}, 0, 0)$, contained in $[\frac{1}{2}, \frac{3}{4}] \times [-\frac{1}{2}, \frac{1}{2}]^2$, plus the straight line from $(\frac{3}{4}, 0, 0)$ to $(1, 0, 0)$. Let $Z = D^3 \cup (A \times I) < R^4$, $X = (0, 0, 0) \times I$ and $Y = \varnothing$. Let $V_t = ([-t, t]^3 \times I) \cap Z$ for $t = \frac{1}{4}$, $t = \frac{3}{4}$. Then $\mathrm{Bdy}_Z V_t$ is bicollared and the projection map $p: V_t \to X$ given by $p(x, y, z, w) = (0, 0, 0, w)$ yields point inverses which are either arcs or the 3-ball $([-t, t]^3 \times 0)$. Obviously $V_{1/4}$ is a regular neighborhood of X in Z. But $V_{3/4}$ is not such a regular neighborhood, since any homeomorphism of $V_{3/4}$ onto $V_{1/4}$ would have to take the knotted ball pair $([-\frac{3}{4}, \frac{3}{4}]^3, A \cap [-\frac{3}{4}, \frac{3}{4}]^3)$ onto the unknotted ball pair $([-\frac{1}{4}, \frac{1}{4}]^3, [-\frac{1}{4}, \frac{1}{4}] \times 0 \times 0)$.

6. The stellar neighborhood theorem.

DEFINITION. If $Y < X < Z$ then *a stellar triangulation* of (Z, X, Y) is a full triangulation $(J, K, L; h)$ such that $J = N(K - L, J)$.

The purpose of this section is to prove

THEOREM 6.1 (THE STELLAR NEIGHBORHOOD THEOREM). *Suppose that* $X, Y, V < Z$ *where V is a topological neighborhood of $X - Y$ and $V \cap Y = Y_R$. Then*

(a) *If there is a stellar triangulation $(J, K, L; h)$ of (V, X_R, Y_R) such that $h(\dot{N}(K - L, J) - L)$ is locally collared in V then there are regular neighborhoods U and U_0 of X mod Y in Z, contained in V, and there is an isotopy H of V into itself such that $H|(V \times 0) \cup (U_0 \times I) = 1$ and $H_1(V) = U$.*

(b) *If $(\mathrm{Bdy}_Z V) - Y$ is locally bicollared in Z and if there is a stellar triangulation $(J, K, L; h)$ of (V, X_R, Y_R) such that $h\dot{N}(K - L, J) = (\mathrm{Bdy}_Z V) \cup Y_R$ then V is a regular neighborhood of X mod Y in Z.*

(c) *If there is a full triangulation $(J, K, L; h)$ of (Z, X, Y) such that $hN(K - L, J) = V$ and $\dot{N}(K - L, J) - L$ is locally collared in $N(K - L, J)$ and in $C(K - L, J)$ then V is a regular neighborhood of X mod Y in Z.*

REMARK 1. Surprisingly, it is not enough to assume in (c) that $S = (\dot{N}(K - L, J) - L)$ is bicollared in J. This is illustrated when $L = \varnothing$, $K = \{a, b\}$ and $J = acd \cup bcd$. The problem is that S may be bicollared and yet not be contained in $(\mathrm{Bdy}_Z V) - Y$. However, it is easy to show that if S is locally bicollared and contained in $(\mathrm{Bdy}_Z V) - Y$ then S is locally collared in $N(K - L, J)$ and in $C(K - L, J)$.

REMARK 2. The proof will be rather long because of technical difficulties resulting when $Y \neq \varnothing$. However, the basic idea of the general proof is given when $Y = \varnothing$ by the following observation: If $K \triangleleft J = N(K, J)$ and if $\dot{N}(K, J)$ is locally collared in J then $N(K, J)'$ is the union of a first derived neighborhood of K and a collar.

We first prove two preliminary lemmas.

LEMMA 6.2. *Suppose that* $f: \sigma \to \tau$ *is a simplicial mapping. Let* τ' *be the standard barycentric subdivision of* τ. *Let* σ' *be the first derived of* σ *determined by the condition that* $b(\alpha)$ *is the centroid of the convex cell* $(f \mid \alpha)^{-1} b(f(\alpha))$, *for each* $\alpha < \sigma$. *Then* $D(\rho, \sigma)$ *is convex for each* $\rho < \sigma$.

Proof. If $\rho = x_0 \cdots x_m$ then $D(\rho, \sigma) = \bigcap \{ D(x_i, \sigma) \mid 0 \leq i \leq m \}$. Therefore it suffices to show that $D(v, \sigma)$ is convex for each vertex v.

Let $\tau = u_0 \cdots u_q$ and let $f^{-1}(u_i) = v_0^i v_1^i \cdots v_{n_i}^i = \sigma_i$. (We may assume that f is onto, so $f^{-1}(u_i) \neq \varnothing$.) Set $\sigma = v_0 \cdots v_n$ where the v's are indexed so that

$$v_j^i = v_{(n_0 + 1) + \cdots + (n_{i-1} + 1) + j} \qquad (n_{-1} \equiv -1).$$

Let $\lambda_k(x)$ or $\lambda_j^i(x)$ be the barycentric coordinate of $x \in \sigma$ with respect to v_k or v_j^i respectively. Notice—or if desired take as definition—the fact that the centroid of $f^{-1} b(\tau)$ is given by

$$[\text{centroid of } f^{-1} b(\tau)] \equiv b(\sigma) = \frac{1}{q+1} \sum_{j=0}^{q} b(\sigma_j)$$

$$= \frac{1}{q+1} \sum_{j=0}^{q} \left(\frac{v_0^j + v_1^j + \cdots + v_{n_j}^j}{n_j + 1} \right).$$

A similar formula holds for the barycenter of each face of σ.

We prove that $D(v_0, \sigma)$ is convex by exhibiting it as the intersection of σ with a family of half-spaces. Through each 1-simplex $v_0 v_j$ of σ we define a hyperplane H_j bisecting $v_0 v_j$. This in turn determines the appropriate half-space H_j^+ containing v_0. If $1 \leq j \leq (n_0 + 1)$ then the defining equations are:

Hyperplane H_j: $\lambda_0 = \lambda_j$.

Half-space H_j^+: $\lambda_0 \geq \lambda_j$.

(NOTE. H_j is the plane generated by $b(v_0 v_j) * \text{Lk } (v_0 v_j, \sigma)$.) If $j > (n_0 + 1)$ and $v_j = v_k^i$ then the appropriate equations are:

Hyperplane H_j: $\lambda_0^0 + \cdots + \lambda_{n_0}^0 = \lambda_0^i + \cdots + \lambda_{n_i}^i$.

Half-space H_j^+: $\lambda_0^0 + \cdots + \lambda_{n_0}^0 \geq \lambda_0^i + \cdots + \lambda_{n_i}^i$.

(NOTE. H_j is the plane generated by [middle slice of $\sigma_0 * \sigma_i$] $* \text{Lk } (\sigma_0 \sigma_i, \sigma)$.)

To show that $|N(v_0, \sigma')|$ is included in the convex set $\bigcap_j (H_j^+ \cap \sigma)$ it suffices to prove that each vertex of $N(v_0, \sigma')$ is in this intersection. This is a straightforward and tedious exercise in barycentric coordinates using the formula for centroids given above. To demonstrate the reverse inclusion one considers a point x in $(\sigma - |N(v_0, \sigma')|)$. The carrier of x in σ' is a simplex of the form $b(\alpha_0) \cdots b(\alpha_s)$ where

$v_0 \nless \alpha_0$ (and, as always, $\alpha_0 < \alpha_1 < \cdots < \alpha_s$). A calculation of barycentric coordinates proves that there is an H_j such that $x \notin H_j^+$,   q.e.d.

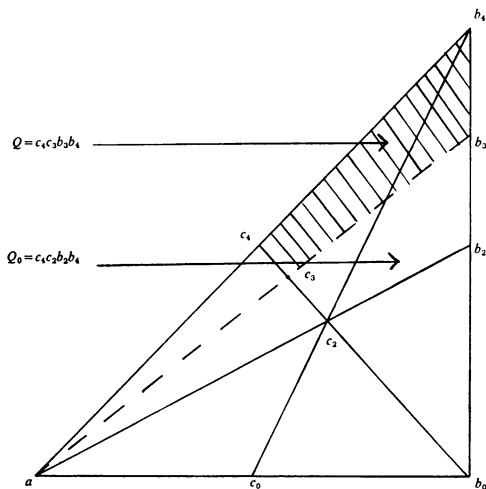LEMMA 6.3. *Assume that* $\Delta = ab_0b_4$ *is the 2-simplex pictured below.*

FIGURE 1

Let Δ' be the barycentric subdivision of Δ. Assume that $\eta: \psi\beta \to \Delta$ is a simplicial mapping such that $b_4 < \eta(\psi) < ab_4$ and $\eta(\beta) < b_0$. (β may be empty.) Let $(\psi\beta)'$ be the first derived of $\psi\beta$ chosen so that $b(\alpha)$ is the centroid of $(\eta|\alpha)^{-1}b(\eta(\alpha))$, for each $\alpha < \psi\beta$. Let $(\psi\beta)^*$ be the first derived of $\psi\beta$ chosen so that the barycenter of α, denoted $b^*(\alpha)$, is defined as follows: (see Figure 1)

  (i) If $\eta(\alpha) < ab_0 \cup ab_4$ then $b^*(\alpha) = b(\alpha)$.

  (ii) If $\alpha = \rho\beta_0$ where $\rho < \psi$, $\varnothing \neq \beta_0 < \beta$ and $b_4 < \eta(\rho)$ then

$$b^*(\alpha) = \eta^{-1}(c_3b_3) \cap b(\rho)b(\rho\beta_0).$$

Let $D(\ ,\ )$ and $D^*(\ ,\ )$ denote duals taken with respect to $(\psi\beta)'$ and $(\psi\beta)^*$ respectively. Finally, let ρ be any simplex of ψ such that $b_4 < \eta(\rho)$. Then:

  (1) $|D^*(\rho, \psi\beta)| = \eta^{-1}(Q) \cap |D(\rho, \psi\beta)|$,

  (2) $|C(D^*(\rho, \psi), D^*(\rho, \psi\beta))| = \eta^{-1}(c_3b_3) \cap |D(\rho, \psi\beta)|$.

Proof. For simplicity we denote

$$C(\rho; \psi, \psi\beta) \equiv C(D(\rho, \psi), D(\rho, \psi\beta)) = C(\dot{D}(\rho, \psi), \dot{D}(\rho, \psi\beta)),$$

$$C^*(\rho; \psi, \psi\beta) \equiv C(D^*(\rho, \psi), D^*(\rho, \psi\beta)) = C(\dot{D}^*(\rho, \psi), \dot{D}^*(\rho, \psi\beta)).$$

The proof is by induction on the codimension of ρ in $\psi\beta$; i.e. on $n = (\dim \psi\beta - \dim \rho)$. The result is trivial when $n = 0$. We assume that $n > 0$ and the result is known for integers less than n.

STEP I. Let $A = b(\rho_0) \cdots b(\rho_s) b^*(\rho_{s+1}\beta_{s+1}) \cdots b^*(\rho_t \beta_t)$ be a simplex of $D^*(\rho, \psi\beta)$ where $\rho < \rho_0 < \cdots < \rho_t < \psi$, and $\varnothing \neq \beta_{s+1} < \cdots < \beta_t < \beta$. Recall that

$$b^*(\rho_j \beta_j) = [\eta^{-1}(c_3 b_3)] \cap [b(\rho_j) b(\rho_j \beta_j)].$$

Thus η takes each vertex of A into Q. Since Q is convex and η is linear on $\psi\beta$ it follows that $\eta(A) \subset Q$. Also it is clear that each vertex of A belongs to $|D(\rho, \psi\beta)|$. The latter set is convex by (6.2) so $A \subset \eta^{-1}(Q) \cap |D(\rho, \psi\beta)|$. In case $s = -1$, i.e. $A < C^*(\rho; \psi, \psi\beta)$, this argument shows that $A \subset \eta^{-1}(c_3 b_3) \cap |D(\rho, \psi\beta)|$. Hence $|D^*(\rho, \psi\beta)| \subset \eta^{-1}(Q) \cap |D(\rho, \psi\beta)|$ and $C^*(\rho; \psi, \psi\beta) \subset \eta^{-1}(c_3 b_3) \cap |D(\rho, \psi\beta)|$.

STEP II. We claim that

(i) $\eta^{-1}(Q) \cap D(\rho, \psi\beta)$ is an n-ball,

(ii) $\eta^{-1}(c_3 b_3) \cap D(\rho, \psi\beta)$ is an $(n-1)$-ball,

(iii) $\partial[(\eta^{-1}Q) \cap D(\rho, \psi\beta)] = [(\eta^{-1}Q) \cap \partial D(\rho, \psi\beta)] \cup [(\eta^{-1}(c_3 b_3)) \cap D(\rho, \psi\beta)],$

(iv) $\partial[(\eta^{-1}(c_3 b_3)) \cap D(\rho, \psi\beta)] = (\eta^{-1}(c_3 b_3)) \cap \partial D(\rho, \psi\beta).$

To see this, notice that there is a first derived Q_0' of Q_0 such that

$$Q = |N((c_4 b_4)', Q_0')| \quad \text{and} \quad c_3 b_3 = |\dot{N}((c_4 b_4)', Q_0')|.$$

If we take a first derived of $D(\rho, \psi\beta)$ such that $\eta | D(\rho, \psi\beta)' : D(\rho, \psi\beta)' \to Q_0'$ is simplicial then it follows from (2.14) that

$$(\eta^{-1}Q) \cap D(\rho, \psi\beta) = N(D(\rho, \psi)', D(\rho, \psi\beta)'),$$

$$(\eta^{-1}(c_3 b_3)) \cap D(\rho, \psi\beta) = \dot{N}(D(\rho, \psi)', D(\rho, \psi\beta)').$$

Then assertions (i)–(iv) are immediate consequences of the convexity of the sets in (i) and (ii) and the basic facts about first derived neighborhoods in a manifold (e.g. (3.4) of [C]).

STEP III. Clearly $D(\rho, \psi\beta)$ is an n-ball. We prove that $C(\rho; \psi, \psi\beta)$ is an $(n-1)$-ball and we calculate $\partial D(\rho, \psi\beta)$ and $\partial C(\rho; \psi, \psi\beta)$ explicitly. Analogous formulas hold for $D^*(\rho, \psi\beta)$ and $C^*(\rho; \psi, \psi\beta)$. Let $\alpha = \mathrm{Lk}\,(\rho, \psi)$. Then

$$D(\rho, \psi\beta) = \{b(\rho\alpha_0\beta_0) \cdots b(\rho\alpha_q\beta_q) \mid \alpha_i < \alpha, \beta_i < \beta\}$$

$$\partial D(\rho, \psi\beta) = \dot{D}(\rho, \psi\beta) \cup D(\rho, \partial(\psi\beta)) \text{ by (5.6) of [C]}$$

$$= \{b(\rho\alpha_0\beta_0) \cdots b(\rho\alpha_q\beta_q) \mid (\beta_0 \neq \varnothing \text{ or } \alpha_0 \neq \varnothing) \text{ or } (\alpha_q \neq \alpha \text{ or } \beta_q \neq \beta)\}$$

$$= C(\rho; \psi, \psi\beta) \cup (\bigcup \{D(\mu, \psi\beta) \mid \rho \lneqq \mu < \psi\})$$

$$\cup (\bigcup \{D(\rho, \bar{\psi}\beta) \mid \rho < \bar{\psi} \lneqq \psi\}) \cup (\bigcup \{D(\rho, \psi\beta_q) \mid \beta_q \lneqq \beta\}).$$

Under the natural isomorphism between $\dot{D}(\rho, \psi\beta)$ and $\mathrm{Lk}\,(\rho, \psi\beta)' = (\alpha\beta)'$, $C(\rho; \psi, \psi\beta)$ goes onto $N(\beta', (\alpha\beta)')$. This proves that $C(\rho; \psi, \psi\beta)$ is an $(n-1)$-ball. Using the fact that $\partial N(\beta', (\alpha\beta)') = \dot{N}(\beta', (\alpha\beta)') \cup N(\beta', \partial(\alpha\beta)')$ we can calculate $\partial N(\beta', (\alpha\beta)')$ just as we calculated $\partial D(\rho, \psi\beta)$ above. Returning under the inverse isomorphism this yields:

$$\partial C(\rho; \psi, \psi\beta) = (\bigcup \{C(\mu; \psi, \psi\beta) \mid \rho \lneqq \mu < \psi\})$$

$$\cup (\bigcup \{C(\rho; \bar{\psi}, \bar{\psi}\beta) \mid \rho < \bar{\psi} \lneqq \psi\}) \cup (\bigcup \{C(\rho; \psi, \psi\beta_q) \mid \beta_q \lneqq \beta\}).$$

STEP IV. We prove that $|C^*(\rho;\psi,\psi\beta)| = (\eta^{-1}(c_3b_3)) \cap |D(\rho,\psi\beta)|$. We have already shown (Step I) that one of these is included in the other and we know (Steps II and III) that each is an $(n-1)$-ball. Thus it suffices to prove that

$$|\partial C^*(\rho;\psi,\psi\beta)| = \partial[\eta^{-1}(c_3b_3) \cap |D(\rho,\psi\beta)|].$$

But

$$\partial C^*(\rho;\psi,\psi\beta) = (\bigcup \{C^*(\mu;\psi,\psi\beta) \mid \rho \lneqq \mu < \psi\}) \cup (\bigcup \{C^*(\rho;\bar\psi,\bar\psi\beta) \mid \rho < \bar\psi \lneqq \psi\})$$

$$\cup (\bigcup \{C^*(\rho;\psi,\psi\beta_q) \mid \beta_q \lneqq \beta\}), \quad \text{by Step III,}$$

$$= \eta^{-1}(c_3b_3) \cap [\bigcup \{D(\mu,\psi\beta) \mid \rho \lneqq \mu < \psi\} \cup (\bigcup \{D(\rho,\bar\psi\beta) \mid \rho < \bar\psi \lneqq \psi\})$$

$$\cup (\bigcup \{D(\rho,\psi\beta_q) \mid \beta_q \lneqq \beta\}) \cup C(\rho;\psi,\psi\beta)],$$

noting that $(\eta^{-1}(c_3b_3)) \cap C(\rho;\psi,\psi\beta) = 0$, and using the induction hypothesis

$$= \eta^{-1}(c_3b_3) \cap \partial D(\rho,\psi\beta) \quad \text{by Step III}$$

$$= \partial[\eta^{-1}(c_3b_3) \cap D(\rho,\psi\beta)] \quad \text{by (iv) of Step II.}$$

This proves assertion (2) of our lemma.

STEP V. To prove that $|D^*(\rho,\psi\beta)| = \eta^{-1}Q \cap |D(\rho,\psi\beta)|$ we note that these are each n-balls and that one is included in the other. Thus it suffices to prove that their boundaries coincide. The argument is similar to Step IV,   q.e.d.

Proof of (6.1c). We shall be working with the simplicial complex J. To simplify the notation we write $V = N(K-L,J)$, $\dot V = \dot N(K-L,J)$ and $W = C(K-L,J)$. We may assume (taking a barycentric subdivision of J modulo V if necessary) that V and $\dot V$ are full in J.

Let $\Delta = ab_0b_4$. (See Figure 1 and (6.3).) Let Δ' be the barycentric subdivision of Δ and let Δ^* be the first derived with vertices a, b_0, b_4, c_0, b_3, c_4, and c_3. As usual $\eta: N(K-L,J) \to \Delta$ is the natural parametrization. Let J' be a first derived of J such that $b(\sigma)$ is the centroid of $(\eta|\sigma)^{-1}b(\eta(\sigma))$, for each simplex $\sigma < N(K-L,J)$. Define J^* as the first derived of J which agrees with J' on $C(K-L,J) \cup K$ and for which $b^*(\rho\beta) = \eta^{-1}(c_3b_3) \cap b(\rho)b(\rho\beta)$ whenever $\rho < (\dot V-L)$, $\varnothing \neq \beta < C(L,K)$ and $\rho\beta < J$. Notice that $\eta: J' \to \Delta'$ and $\eta: J^* \to \Delta^*$ are simplicial mappings. Simplicial structures relating to J' are denoted as usual, while those relating to J^* are denoted by asterisks. If $\rho < S < T < J$ then $C(\rho;S,T) \equiv C(D(\rho,S),D(\rho,T))$ and $C^*(\rho;S,T) \equiv C(D^*(\rho,S),D^*(\rho,T))$. The symbol ρ always denotes a variable nonempty simplex of $\dot V-L$. Finally, we set

$$\mathcal{N} = N(\dot V-L,J'), \qquad \dot{\mathcal{N}} = \dot N(\dot V-L,J'),$$

$$\mathcal{N}_v = N(\dot V-L,V'), \qquad \dot{\mathcal{N}}_v = \dot N(\dot V-L,V'),$$

$$\mathcal{N}_v^* = N(\dot V-L,V^*), \qquad \dot{\mathcal{N}}_v^* = \dot N(\dot V-L,V^*).$$

(See Figure 2.)

FIGURE 2

STEP I. We apply the theorem on collared subpolyhedra (4.2D) to the triples $(V, \dot{V}, V \cap L)$ and $(W, \dot{V}, L)$. Thus there is a homeomorphism $g: C(L', \mathcal{N}) \to C(L', \dot{V}') \times [0, 3]$ such that, for each simplex $\rho < \dot{V} - L$,

$$D(\rho, W) \to D(\rho, \dot{V}) \times [2, 3],$$

$$D(\rho, V) \to D(\rho, \dot{V}) \times [0, 2],$$

$$D(\rho, \dot{V}) \overset{1}{\to} D(\rho, \dot{V}) \times 2,$$

$$C(\rho; \dot{V}, W) \to D(\rho, \dot{V}) \times 3,$$

$$C(\rho; \dot{V}, V) \to D(\rho, \dot{V}) \times 0.$$

Claim. g can be chosen so as to further satisfy

$$C(L', \mathcal{N}_V^*) \to C(L', \dot{V}') \times [1, 2],$$

$$D^*(\rho, V) \to D(\rho, \dot{V}) \times [1, 2],$$

$$C^*(\rho; \dot{V}, V) \to D(\rho, \dot{V}) \times 1.$$

Lemma used to prove the claim:

(1) $\eta^{-1}(Q) = C(L', \mathcal{N}_V^*)$ and $\eta^{-1}(c_3b_3) = C(L', \dot{\mathcal{N}}_V^*)$.

(2) If $\rho < \dot{V} - L$ then

$$\eta^{-1}(Q) \cap D(\rho, V) = D^*(\rho, V) \text{ and } \eta^{-1}(c_3b_3) \cap D(\rho, V) = C^*(\rho; \dot{V}, V).$$

(3) If $\rho < \psi < \dot{V} - L$, $\rho < \dot{V} - L$, $\psi\beta < V$, and $\beta < C(L, K)$, then

$$\eta^{-1}(Q) \cap D(\rho, \psi\beta) = D^*(\rho, \psi\beta) \text{ and } \eta^{-1}(c_3b_3) \cap D(\rho, \psi\beta) = C^*(\rho; \psi, \psi\beta).$$

Proof of the lemma. By taking successive unions one proves easily that (3) $\Rightarrow$ (2) $\Rightarrow$ (1). But (3) is precisely the conclusion of (6.3).

Proof of the claim. $\eta | C(L', \mathcal{N}_v)$ is a simplicial mapping into Q_0. There is a first derived Q_0' of Q_0 such that

$$Q = |N(c_4b_4 - c_2b_2, Q_0')| \text{ and } c_3b_3 = |\dot{N}(c_4b_4 - c_2b_2, Q_0')|.$$

We choose a first derived $C(L', \mathcal{N}_v)'$ such that $\eta: C(L', \mathcal{N}_v)' \to Q_0'$ is also simplicial. Then, by (2.14), $(\eta^{-1}(Q), \{\eta^{-1}(Q) \cap D(\rho, V)\})$ is a regular neighborhood of $C(L', \dot{V}')$ mod $C(L', \dot{\mathcal{N}}_v)$ in $(C(L', \mathcal{N}_v), \{D(\rho, V)\})$. Thus by the lemma, $(C(L', \mathcal{N}_V^*), \{D^*(\rho, V)\})$ is such a regular neighborhood. Notice also that the boundary of this regular neighborhood is $(C(L', \dot{\mathcal{N}}_V^*), \{C^*(\rho; \dot{V}, V)\})$.

From the above we see that $g(C(L', \mathcal{N}_V^*), \{D^*(\rho, V)\})$ is a regular neighborhood of $C(L', \dot{V}') \times 2$ mod $C(L', \dot{V}') \times 0$ in $(C(L', \dot{V}') \times [0, 2], \{D(\rho, \dot{V}) \times [0, 2]\})$. Therefore we may apply Addendum 3.3 of the uniqueness theorem to get a homeomorphism g_1 of $(C(L', \dot{V}') \times [0, 2], \{D(\rho, \dot{V}) \times [0, 2]\})$ onto itself such that

$$g_1 | C(L', \dot{V}') \times \{0, 2\} = 1,$$

$$g_1 g C(L', \mathcal{N}_V^*) = C(L', \dot{V}') \times [1, 2],$$

$$g_1 g D^*(\rho, V) = D(\rho, \dot{V}) \times [1, 2],$$

$$g_1 g C^*(\rho; \dot{V}, V) = D(\rho, \dot{V}) \times 1.$$

If we define g_1 to be the identity on $C(L', \dot{V}') \times [2, 3]$ then $g_1 g = g_2$ is a homeomorphism which satisfies the claim.

STEP II. We now define a homeomorphism h_0 of J onto J such that $h_0 | C(\dot{V} - L, J') = 1$ and $h_0(V) = N(K - L, J^*)$. Since $(K' \cup L') < C(\dot{V} - L, J')$, h_0 is the identity on $(K' \cup L')$. (For the rest of this paragraph let $V = hN(K - L, J)$.) Hence $(J, K, L; hh_0^{-1})$ is a full triangulation of (Z, X, Y) such that

$$V = hh_0^{-1}N(K - L, J^*).$$

This proves that V is a regular neighborhood of X mod Y in Z.

Let $u: [0, 3] \to [0, 3]$ be a homeomorphism such that $u(0) = 0$, $u(2) = 1$ and $u(3) = 3$. Let $k = 1 \times u: C(L', \dot{V}') \times [0, 3] \to C(L', \dot{V}') \times [0, 3]$. Let g be the homeomorphism constructed in Step I. Then $g^{-1}kg = h_1$ is a homeomorphism of $C(L', \mathcal{N})$

onto itself. Using the properties of g we see that

(II.1) $h_1 D(\rho, J) = D(\rho, J)$ for each $\rho < \dot{V} - L$,

(II.2) $h_1 | C(\rho; \dot{V}, J) = 1$,

(II.3) $h_1 D(\rho, \dot{V}) = C^*(\rho; \dot{V}, V)$,

(II.4) $h_1 D(\rho, W) = D^*(\rho, J)$.

Let $L_0 = L \cap N(\dot{V} - L, \dot{V})$. Then, by (2.10),

$$\begin{aligned}
\mathcal{N} &= N(\dot{V} - L, J') \\
&= \{b(\alpha_0 \gamma_0 \delta_0) \cdots b(\alpha_q \gamma_q \delta_q) \mid \alpha_i \gamma_i \delta_i < N(\dot{V} - L, J), \gamma_i = \varnothing \Rightarrow \delta_i = \varnothing\} \\
&= \bigcup \{A D(\rho, J) | \rho < (\dot{V} - L) \text{ and:} \\
&\qquad\qquad A = \varnothing \text{ or } A = b(\alpha_0) \cdots b(\alpha_r) < L_0' \text{ with } \alpha_r < \rho\}
\end{aligned}$$

$$\dot{\mathcal{N}} = \bigcup \{A C(\rho; \dot{V}, J) | \rho < (\dot{V} - L) \text{ and:} \\
\qquad\qquad A = \varnothing \text{ or } A = b(\alpha_0) \cdots b(\alpha_r) < L_0' \text{ with } \alpha_r < \rho\}$$

$$C(L', \mathcal{N}) = \bigcup \{D(\rho, J) | \rho < (\dot{V} - L)\}$$

$$C(L', \dot{\mathcal{N}}) = \bigcup \{C(\rho; \dot{V}, J) | \rho < (\dot{V} - L)\}.$$

Extend h_1 to a mapping $h_2: \mathcal{N} \to \mathcal{N}$ by stipulating that

$$h_2 | [A * D(\rho, J)] = 1_A * h_1 | D(\rho, J).$$

By (II.1), $h_2 | A D(\rho, J)$ is a homeomorphism of $A D(\rho, J)$ onto itself. Also

$$(A_0 D(\rho_0, J)) \cap (A_1 D(\rho_1, J)) = (A_0 \cap A_1) D(\rho_0 \cdot \rho_1, J),$$

where $\rho_0 \cdot \rho_1$ is the smallest simplex of J containing both ρ_0 and ρ_1, and this dual is understood to be empty if there is no such simplex. Therefore h_2 is a well-defined homeomorphism of $\mathcal{N}$ with itself. From (II.2) it is clear that

$$h_2 | \dot{\mathcal{N}} = h_2 | \dot{N}(\dot{V} - L, J') = 1.$$

So h_2 extends to a homeomorphism h_0 of J which is the identity on $C(\dot{V} - L, J')$. It remains only to show that $h_0(V) = N(K - L, J^*)$.

Suppose that $x \in J - V$. If $h_0(x) = x$ then $h_0(x) \notin N(K - L, J^*)$. If $h_0(x) \neq x$ then $x \in [A D(\rho, W) - A D(\rho, \dot{V})]$ for some simplexes $A < L_0$ and $\varnothing \neq \rho < \dot{V} - L$. Therefore, by (II.3) and (II.4), $h_0(x) \in [A D^*(\rho, J) - A C^*(\rho; \dot{V}, V)]$. Hence the carrier of $h_0(x)$ in J^* is of the form $B = b(\alpha_0) \cdots b(\alpha_q) b^*(\rho_0) b^*(\rho_1 \beta_1) \cdots b^*(\rho_r \beta_r)$, where $\rho < \rho_0 < \dot{V} - L$. Thus $B \nless N(K - L, J^*)$, and we see that, in any case, $x \notin V$ implies $h_0(x) \notin N(K - L, J^*)$. Therefore $N(K - L, J^*) < h_0(V)$.

To prove the opposite inclusion notice that, on the one hand, $h_0 N(K - L, J') = N(K - L, J') = \eta^{-1}(ab_0 b_2) < \eta^{-1}(ab_0 b_3) = N(K - L, J^*)$. On the other hand, if B is a simplex of $N(K - L, J)' - N(K - L, J')$ then

$$B = b(\alpha_0) \cdots b(\alpha_q) b(\alpha_{q+1} \gamma_{q+1}) b(\alpha_{q+1} \beta_{q+2} \gamma_{q+2}) \cdots b(\alpha_r \beta_r \gamma_r)$$

where $\alpha_i \beta_i \gamma_i < N(K - L, J)$ and $\varnothing \neq \gamma_{q+1} < \dot{N}(K_R, N(K - L, J))$. Let $\rho = \alpha_{q+1} \gamma_{q+1}$.

Then $\rho < (\dot{V} - L)$ because $V \cap L = L_R$. Thus $B < AD(\rho, V)$ where $A < L'_0$ and $\rho < \dot{V} - L$. Hence

$$h_0(B) < A * h_0 D(\rho, V)$$
$$= A * h_0(\mathrm{Cl}\,[D(\rho, J) - D(\rho, W)])$$
$$= A * \mathrm{Cl}\,[D(\rho, J) - D^*(\rho, J)], \qquad \text{by (II.1), (II.4).}$$

Let P denote the closure in the last expression. Now $D^*(\rho, J) > D(\rho, J) \cap \eta^{-1} Q$ by the lemma in Step I; so if $x \in P$ then $\eta(x) \in ab_0 b_3$. Since $A < L'_R$, $\eta(A) = a$. Hence, by linearity of η, $\eta(A * P) < ab_0 b_3$. Thus

$$h_0(B) < A * P < \eta^{-1}(ab_0 b_3) = N(K - L, J^*).$$

Therefore $h_0(V) = N(K - L, J^*)$. The proof of (6.1c) is complete,   q.e.d.

Proof of (6.1b). We assume that $(\mathrm{Bdy}_Z V) - Y$ is locally bicollared in Z and that there is a stellar triangulation $(J, K, L; h)$ of (V, X_R, Y_R) with $h\dot{N}(K - L, J) = (\mathrm{Bdy}_Z V) \cup Y_R$.

Since

$$(\mathrm{Bdy}_Z V) \cup Y_R < \mathrm{Cl}\,((Z - V)) \cup Y_R,$$

it follows from the definition of a subpolyhedron that there is a partition $(\dot{N}(K - L, J))_*$ and a triangulation

$$(Q, \dot{N}(K - L, J)_*, K_0, L_0; q)$$

of

$$(\mathrm{Cl}\,((Z - V)) \cup Y_R, (\mathrm{Bdy}_Z V) \cup Y_R, \mathrm{Cl}\,((X - V)) \cup Y_R, \mathrm{Cl}\,((Y - V)) \cup Y_R = Y)$$

such that

$$q \mid |\dot{N}(K - L, J)_*| = h \mid |\dot{N}(K - L, J)|$$

and

$$|Q| \cap |J| = q^{-1}((\mathrm{Bdy}_Z V) \cup Y_R) = |\dot{N}(K - L, J)|.$$

We may assume that each of the subcomplexes mentioned is full in Q.

Extend the partition $\dot{N}(K - L, J)_*$ to a partition J_* of $J = N(K - L, J)$ by demanding that $(\alpha\beta\gamma)_* = \beta((\alpha\gamma)_*)$ for all $\alpha\beta\gamma < J$. This gives a well-defined triangulation of $|J_*| \cup |Q|$ because $|J_*| \cap |Q|$ inherits the same triangulation from J_* and from Q. Let $J_1 = J_* \cup Q$, $K_1 = K_* \cup K_0$ and $L_1 = L_0$. Define $h_1 : J_1 \to Z$ by the condition that $h_1|J_* = h$ and $h_1|Q = q$. One checks easily that $(J_1, K_1, L_1; h_1)$ is a full triangulation of (Z, X, Y). Also, notice that $N(K - L, J)_* = N(K_1 - L_1, J_1)$ and $\dot{N}(K - L, J)_* = \dot{N}(K_1 - L_1, J_1)$. Therefore $V = h_1 N(K_1 - L_1, J_1)$. Finally,

$$h_1(\dot{N}(K_1 - L_1, J_1) - L_1) = h(\dot{N}(K - L, J) - L) = (\mathrm{Bdy}_Z V) - Y.$$

Hence $\dot{N}(K_1 - L_1, J_1) - L_1$ is locally bicollared in J_1. So, since this set lies in the topological boundary of $N(K_1 - L_1, J_1) - L_1$, it is locally collared in $N(K_1 - L_1, J_1)$ and in $C(K_1 - L_1, J_1)$. The hypotheses of (6.1c) are satisfied. Therefore V is a regular neighborhood of $X \bmod Y$ in Z,   q.e.d.

Proof of (6.1a). We are given a stellar triangulation $(J, K, L; h)$ of (V, X_R, Y_R) such that $h(\dot{N}(K-L, J)-L)$ is locally collared in V. Let J' and J^* be first deriveds of J chosen as in the proof of (6.1c). Let $V_1 = N(K-L, J^*)$ and $V_0 = N(K-L, J')$. Clearly V_1 is a regular neighborhood of X_R mod Y_R in V. The argument for (6.1c), restricted to $N(K-L, J)$, can be easily modified to yield an isotopy G of V into V such that $G_0 = 1$, $G_t | V_0 = 1$ and $G_1(V) = V_1$. Some subdivision $(J_0, K_0, L_0; h)$ of $(J', K', L'; h)$ extends to a full triangulation $(J_1, K_1, L_1; h_1)$ of (Z, X, Y). Let $U = h_1 N(K_1 - L_1, J'_1)$ and let $U_0 = h_1 N(K_1 - L_1, J''_1)$. These are both regular neighborhoods of X mod Y in Z and of X_R mod Y_R in V. Then, by Addendum (3.2) (which will be proved in the next section, using (6.1c) and (6.1b)) there is an isotopy F of V onto itself such that $F_0 = 1$, $F_t | U_0 = 1$ and $F_1(V_1) = U$. Thus $H = FG$ is an isotopy of V into V with the desired properties,   q.e.d.

7. **Relationships between regular neighborhoods.** Suppose that X, Y, $Z_0 < Z$ and X_0, $Y_0 < Z_0$. In this section we answer the following practical questions:

What is the effect on a regular neighborhood of increasing or decreasing the polyhedron Y?

What are the relationships between two regular neighborhoods of X mod Y in Z when one contains the other?

When can a regular neighborhood V_0, of $X \cap Z_0$ mod $Y \cap Z_0$ in Z_0, be extended to a regular neighborhood (V, V_0) of X mod Y in (Z, Z_0)?

What is the relationship between (V, V_0) being a regular neighborhood of X mod Y in (Z, Z_0) and V being a regular neighborhood of

$$(X \cup V_0) \text{ mod } (Y \cup \text{Bdy}_{Z_0} V_0)$$

in Z?

Given two regular neighborhoods of X_0 mod Y_0 in Z_0, when can an isotopy of (Z, Z_0) be found which takes one onto the other and keeps fixed a crucial subpolyhedron of Z?

PROPOSITION 7.1.

(a) *If V is a regular neighborhood of X mod Y in Z, $P < Z$, and $(P \cap V) < Y$ then V is a regular neighborhood of X mod $(Y \cup P)$ in Z.*

(b) *If V is a regular neighborhood of X mod $(Y \cup P)$ in Z and if $(P \cap X) < Y$ then V is a regular neighborhood of X mod Y in Z.*

Proof. In both (a) and (b)

$$(X - Y) = X - (Y \cup P),$$

$$Y_R = (Y \cup P)_R = (V \cap Y) = V \cap (Y \cup P),$$

and

$$(\text{Bdy}_Z V) - Y = (\text{Bdy}_Z V) - (Y \cup P).$$

The result then follows in each case because the hypotheses in the stellar neighborhood theorem (6.1b) do not depend on what is happening outside V,   q.e.d.

The analogue of (7.1) for pairs—i.e., for regular neighborhoods (V, V_1) of $X \bmod Y$ (or of $X \bmod Y \cup P$) in (Z, Z_1)— would follow from the stellar neighborhood theorem for pairs. The latter theorem appears to be true by a proof completely analogous to that of (6.1). Since however we have not developed the theory for pairs explicitly, we give a direct proof of that special case of (7.1) for pairs which will be needed in the sequel.

LEMMA 7.2. *Suppose that* $K, L \triangleleft J_0 \triangleleft J$ *and that* P *is a subpolyhedron of* $|J|$ *such that* $P \cap |N(K-L, J)| < |L|$. *Then* $(N(K-L, J'), N(K-L, J_0'))$ *is a regular neighborhood of* $K \bmod (L \cup P)$ *in* (J, J_0).

Proof. There is a partition J_* of J such that $|L| \cup P$ is the underlying space of a full subcomplex, say Q, of J_*. By definition the pair $(N(K-Q, J_*'), N(K-Q, J_{0*}'))$ is a regular neighborhood of $K \bmod (L \cup P)$ in (J, J_0). But this pair is just

$$(N(K-L, J_*'), N(K-L, J_{0*}'))$$

since $P \cap |N(K-L, J)| < |L|$. By the proof of the uniqueness theorem, there is a homeomorphism fixed on $[K \cup C(K-L, J)]$ which takes this pair onto $(N(K-L, J'), N(K-L, J_0'))$. The prescribed homeomorphism obviously is fixed on $K \cup L \cup P$. Therefore $(N(K-L, J'), N(K-L, J_0'))$ is a regular neighborhood of $K \bmod (L \cup P)$ in (J, J_0),   q.e.d.

PROPOSITION 7.3. *If* V *and* V_0 *are regular neighborhoods of* $X \bmod Y$ *in* Z *and* $(V_0 - Y) \subset \operatorname{Int}_Z V$ *then*
  (a) V *is a regular neighborhood of* $V_0 \bmod Y$ *in* Z;
  (b) $\operatorname{Cl}(V - X)$ *is a regular neighborhood of* $V_0 \bmod (X \cup Y)$ *in* Z.

Proof. To prove (a), let $(J, K, L; h)$ be a full triangulation of (Z, X, Y) exhibiting V as a regular neighborhood, $V = hN(K-L, J)$. Let $V_1 = hN(K-L, J'')$. The stellar neighborhood theorem (6.1b) implies that V is a regular neighborhood of $V_1 \bmod Y$ in Z. But (7.1a) implies that V_0 and V_1 are both regular neighborhoods of $X \bmod (\operatorname{Cl}(Z-V)) \cup Y$. Hence there is a homeomorphism g of Z onto itself such that $g(V) = V$, $g|Y = 1$ and $g(V_1) = V_0$. Therefore V is a regular neighborhood of $V_0 \bmod Y$ in Z.

To prove (b), choose a full triangulation $(J, K_0, L; h)$ of (Z, V_0, Y) such that $h^{-1}(X)$ and $h^{-1}(X \cup Y)$ underlie full subcomplexes K and $K \cup L$ of J. By part (a) there is a homeomorphism g of $hN(K_0-L, J')$ onto V such that $g|(V_0 \cup Y) = 1$. Thus $(J, K_0, K \cup L, gh)$ is a full triangulation of $(Z, V_0, X \cup Y)$ such that $gh(N(K_0-L, J')) = V$. Therefore

$$gh(\operatorname{Cl}[N(K_0-L, J') - K]) = \operatorname{Cl}(V - X).$$

However, $\operatorname{Cl}[N(K_0-L, J') - K] = N(K_0-(K \cup L), J')$. Hence $\operatorname{Cl}(V - X)$ is a regular neighborhood of $V_0 \bmod X \cup Y$ in Z,   q.e.d.

REMARK. Addendum (3.2) follows directly from (7.1), (7.3) and the uniqueness theorem for regular neighborhoods.

PROPOSITION 7.4. *If V_0 is a regular neighborhood of X mod Y in Z and V is a regular neighborhood of V_0 mod Y in Z then V is a regular neighborhood of X mod Y in Z.*

Proof. Let U be a regular neighborhood of X mod Y in Z such that $U - Y$ $\subset \mathrm{Int}_Z V_0$. Since $(\mathrm{Cl}\,(Z - V)) \cap U < (\mathrm{Cl}\,(Z - V)) \cap V_0 < Y$ it follows (3.2) that there is a homeomorphism $g: Z \to Z$ such that $g|\,Y = 1$, $g(V) = V$ and $g(V_0) = U$. Hence the hypothesis implies that V is a regular neighborhood of U mod Y in Z. But, by (7.3), V_0 is a regular neighborhood of U mod Y in Z. Therefore there is a homeomorphism h of Z onto itself such that $h(V_0) = V$ and $h|(U \cup Y) = 1$. Then, since V_0 is a regular neighborhood of X mod Y in Z, so is V,   q.e.d.

PROPOSITION 7.5. *Suppose that U and V are regular neighborhoods of X mod Y in Z with $(U - Y) \subset \mathrm{Int}_Z V$. Let $Y_0 = Y \cap \mathrm{Bdy}_Z V$. Then $\mathrm{Cl}\,(V - U)$ is homeomorphic to the regular neighborhood of $(\mathrm{Bdy}_Z V) \times 0$ mod $(Y_0 \times 0)$ in $(\mathrm{Bdy}_Z V) \times I$.*

Proof. Let $Z_0 = V \cup (\mathrm{Bdy}_Z V \times I)$ be the disjoint union with the identification $x = (x, 0)$ if $x \in \mathrm{Bdy}_Z V$. It is obvious from the stellar neighborhood theorem (6.1b) that U and V are regular neighborhoods of X_R mod Y_R in Z_0. Let W be a regular neighborhood of V mod Y_R in Z_0. By checking a triangulation in which W is a first derived neighborhood of V, one sees that $\mathrm{Cl}\,(W - V)$ is a regular neighborhood of $\mathrm{Bdy}_Z V$ mod Y_0 in $(\mathrm{Bdy}_Z V) \times I$. Thus it suffices to exhibit a homeomorphism of Z_0 onto itself which takes V onto W and U onto V.

By (7.3), V is a regular neighborhood of U mod Y_R in Z_0. Hence, by (7.4) W is such a regular neighborhood. Therefore there is a homeomorphism f of Z_0 onto itself such that $f|U = 1$ and $f(V) = W$. Now U and V are regular neighborhoods of X_R mod $(Y_R \cup \mathrm{Cl}\,(Z_0 - W))$ in Z_0, by (7.1). Hence there is a homeomorphism g of Z_0 onto itself such that $g(W) = W$ and $g(U) = V$. Then $gf(V, U) = g(W, U) = (W, V)$, q.e.d.

The following lemma is a special case of (7.9) and will be used to prove (7.9).

LEMMA 7.6. *Suppose that V_0 is a regular neighborhood of X mod Y in Z_0 where $X, Y < Z_0 < Z$ and every triangulation of Z_0 extends to a triangulation of Z. Then, if $P < Z$ and $(P \cap Z_0) < Y$, there is a subpolyhedron V of Z such that (V, V_0) is a regular neighborhood of X mod $(Y \cup P)$ in (Z, Z_0).*

REMARK. Every triangulation of Z_0 extends to one of Z if, for example, Z is a manifold and $Z_0 = \partial Z$. Also (see [A] and [L]) every triangulation extends if (Z, Z_0) is a proper locally unknotted manifold pair or[3] if Z is a manifold and Z_0 is a compact, codimension 3 subpolyhedron in the interior of Z. (In the former case, C. P. Rourke has given an alternate proof, using block bundle theory, which

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[3] The author is grateful to M. A. Armstrong for pointing this out.

eliminates Armstrong's compactness assumption.) Most generally, E. C. Zeeman has asserted that every triangulation extends if and only if the intrinsic i-skeleton of Z_0 equals the ambient intrinsic i-skeleton of Z_0 in Z, for each integer i. However the proof of this assertion seems to be in doubt[4]. [These comments are just background. We shall not use the results alluded to above.]

Proof. Choose a full triangulation $(J_0, K, L; h_0)$ of (Z_0, X, Y) such that $V_0 = h_0 N(K-L, J_0')$. By hypothesis there is a triangulation (J, J_0, h) of (Z, Z_0) such that $h|J_0 = h_0$. We may assume that J_0 is full in J. It would suffice to know that $h^{-1}(P) \cap N(J_0-L, J) < L$. For this implies that $h^{-1}(P) \cap N(K-L, J) < L$. Then, taking a first derived J' of J which agrees on J_0 with J_0', we could apply (7.2) to conclude that $(N(K-L, J'), N(K-L, J_0'))$ is a regular neighborhood of $K \bmod (L \cup h^{-1}P)$ in (J, J_0). It would follow that the image (V, V_0) of this pair under h is a regular neighborhood of $X \bmod (Y \cup P)$ in (Z, Z_0).

Unfortunately it might not occur that $h^{-1}(P) \cap N(J_0-L, J) < L$. However, let $\tilde{J}$ be a first barycentric subdivision of J modulo J_0 and let $g : J \to J$ be a homeomorphism such that

(1) $g|J_0 = 1$,

(2) $h^{-1}(P) \cap gN(J_0-L, \tilde{J}) < L$.

Then $(\tilde{J}, J_0; hg)$ is a triangulation of (Z, Z_0) with all the desired properties. We shall show that such a homeomorphism g exists.

Notice that $|N(J_0-L, \tilde{J})| = |N(J_0-L, J')|$ and interpret (2) accordingly.

g is defined to be the identity on $J_0 \cup C(J_0-L, J)$. Suppose that g has been extended inductively over

$$J_i = J_0 \cup C(J_0-L, J) \cup (i\text{-skeleton of } J)$$

so that, for each $\sigma < J_i$ we have

(a) $g(\sigma) = \sigma$,

(b) $h^{-1}(P) \cap gN(\sigma \cap J_0 - \sigma \cap L, \sigma') < L$.

We extend g over each simplex $\tau < J_{i+1} - J_i$. Since $\tau < N(J_0-L, J) - J_i$, $\tau = \alpha\beta\gamma$ where $\alpha < L$, $\beta < C(L, J_0)$, $\gamma < C(J_0, J)$, $\beta \neq \varnothing$, and $\gamma \neq \varnothing$. By hypothesis

$$[h^{-1}(P) \cap \tau \cap J_0] < |\alpha|.$$

By inductive assumption

$$h^{-1}(P) \cap gN(\dot{\tau} \cap J_0 - \dot{\tau} \cap L, \dot{\tau}') = h^{-1}(P) \cap gN(\alpha\beta-\alpha, \dot{\tau}') < |\alpha|.$$

Let g_τ be a homeomorphism of τ onto τ, fixed on $\dot{\tau}$, which takes

$$b(\tau) * gN(\alpha\beta-\alpha, \dot{\tau}') = (1_{b(\tau)} * g|\dot{\tau})(N(\alpha\beta-\alpha, \tau'))$$

onto a regular neighborhood W of $gN(\alpha\beta-\alpha, \dot{\tau}') \bmod [h^{-1}(P) \cup gC(\alpha\beta-\alpha, \dot{\tau}')]$ in τ. This is possible because, on the one hand, $b(\tau) * gN(\alpha\beta-\alpha, \dot{\tau}')$ is a regular

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neighborhood of $gN(\alpha\beta-\alpha, \dot{\tau}')$ mod $gC(\alpha\beta-\alpha, \dot{\tau}')$ by the stellar neighborhood theorem. On the other hand W is such a regular neighborhood because

$$[h^{-1}(P) \cap gN(\alpha\beta-\alpha, \dot{\tau}')] < |\alpha| < gC(\alpha\beta-\alpha, \dot{\tau}')$$

and we may apply (7.1). Notice that $g_\tau|\dot{\tau}=1$ and $W \cap h^{-1}(P) < |\alpha|$.

Now we extend g over each $\tau < J_{i+1}-J_i$ by defining

$$g|\tau = g_\tau \circ (1_{b(\alpha)} * g|\dot{\tau}).$$

Continuing in this way we define g on all of J. It is easy to see that g has properties (1) and (2),   q.e.d.

PROPOSITION 7.7. *Suppose that V_0 and W_0 are regular neighborhoods of X mod Y in Z_0, where $X, Y < Z_0 < Z$ and every triangulation of Z_0 extends to a triangulation of Z. Then, if $P < Z$ and $(P \cap Z_0) < (X \cup Y)$, there is an isotopy $H_t: (Z, Z_0) \to (Z, Z_0)$ such that*

(1) $H_0 = 1$; $H_t|(X \cup Y \cup P) = 1$ $(0 \leq t \leq 1)$.

(2) $H_1(V_0) = W_0$.

Proof. Let U_0 be a regular neighborhood of X mod Y in Z_0 such that $(U_0 - Y)$ $\subset [\text{Int}_{Z_0}(V_0) \cap \text{Int}_{Z_0}(W_0)]$. Then $\text{Cl}(V_0 - X)$ and $\text{Cl}(W_0 - X)$ are regular neighborhoods of U_0 mod $(X \cup Y)$ in Z_0, using (7.3). Because $(P \cap Z_0) < (X \cup Y)$, the preceding lemma implies that there exist V and W such that (V, V_0) and (W, W_0) are regular neighborhoods of U_0 mod $(X \cup Y \cup P)$ in (Z, Z_0). Therefore there is an isotopy H_t of (Z, Z_0) onto (Z, Z_0) such that $H_0 = 1$, $H_t|(X \cup Y \cup P) = 1$ and $H_1(\text{Cl}(V_0 - X)) = \text{Cl}(W_0 - X)$. Then

$$H_1(V_0) = H_1(\text{Cl}(V_0 - X)) \cup X_R = \text{Cl}(W_0 - X) \cup X_R = W_0,   \text{q.e.d.}$$

There are important situations (some of which arise in §9) where we wish to know the conclusion of (7.7) but do not know that every triangulation of Z_0 extends to one of Z. One such circumstance is covered by

PROPOSITION 7.8. *Suppose that V_0 and W_0 are regular neighborhoods of X mod Y in Z_0 where $X, Y < Z_0 < Z$. Suppose further that $P < Z$, $(Z-P)$ is a manifold, $(Z_0 - P)$ is an open subset of $\partial(Z-P)$, and $(P \cap Z_0) < (X \cup Y)$. Then there is a homeomorphism $G: (Z, Z_0) \to (Z, Z_0)$ such that $G|(X \cup Y \cup P) = 1$ and $G(V_0) = W_0$.*

REMARK. As in (7.7) the homeomorphism we construct is isotopic to the identity.

Proof. Let $g: Z_0 \to Z_0$ be a homeomorphism fixed on $(X \cup Y)$ such that $g(V_0) = W_0$. We may assume, by (3.4), that $g = g_1 g_2 \cdots g_q$ where each g_j is an $(n-1)$-move relative to $(X \cup Y)$. For a given g_j, let $\{Q_i^{n-1}\}$ be the family of balls with disjoint interiors on which g_j is supported, where

$$(Q_i^{n-1} \cap P) < (Q_i^{n-1} \cap (X \cup Y)) < \partial Q_i^{n-1}.$$

By the definition of a move we may choose a single triangulation of Z_0—and then one of Z—in which P, X, Y, and all the Q_i^{n-1} are subcomplexes. Let U be a

relative second derived neighborhood of $\bigcup Q_i^{n-1} \bmod (P \cup \bigcup \partial Q_i^{n-1})$ in Z. Then $U = \bigcup Q_i^n$ where Q_i^n is a regular neighborhood of $Q_i^{n-1} \bmod (P \cup \partial Q_i^{n-1})$ in Z. By construction Q_i^n and Q_k^n have disjoint interiors if $i \neq k$.

We claim that Q_i^n is an n-ball. For, let $Z_1 = Z \cup vQ_i^{n-1}$. Then Int Q_i^{n-1} is locally bicollared in Z_1 since $(Z_0 - P)$ is an open subset of $\partial(Z - P)$. Hence it follows from the stellar neighborhood theorem and (7.4) that vQ_i^{n-1} and $(vQ_i^{n-1} \cup Q_i^n)$ are regular neighborhoods of $v(\partial Q_i^{n-1}) \bmod (\partial Q_i^{n-1})$ in Z_1. But the closure of the difference of these regular neighborhoods is precisely Q_i^n. Thus (7.5) implies that Q_i^n is homeomorphic to a regular neighborhood of $(\sigma^{n-1} \times 0) \bmod (\dot{\sigma}^{n-1} \times 0)$ in $\sigma^{n-1} \times I$. (Here we are using the fact that $\mathrm{Bdy}_{Z_1} (Q_i^n \cup vQ_i^{n-1}) \simeq \mathrm{Bdy}_{Z_1} (vQ_i^{n-1}) = Q_i^{n-1} \simeq \sigma^{n-1}$.) But, choosing a triangulation of $(\sigma^{n-1} \times I)$ in which $\sigma = (\sigma^{n-1} \times 0)$ is a simplex whose link is the vertex w, we have $N(\sigma - \dot{\sigma}, (\sigma \times I)') = b(w\sigma)\sigma'$. Thus Q_i^n is an n-ball.

Now Q_i^{n-1} is a face of Q_i^n and $g_j | Q_i^{n-1}$ is a homeomorphism of Q_i^{n-1} onto itself which is the identity on ∂Q_i^{n-1}. Hence $g_j | Q_i^{n-1}$ extends to a homeomorphism $G_j : Q_i^n \to Q_i^n$ such that $g_j | (\partial Q_i^n - Q_i^{n-1}) = 1$. This defines G_j on $\bigcup Q_i^n$, and we extend G_j to all of Z by defining $G_j | (Z - U) = 1$. The homeomorphism $G = G_1 G_2 \cdots G_q$ has the desired properties,   q.e.d.

PROPOSITION 7.9. *Assume that X, Y, $Z_0 < Z$ and that V_0 is a regular neighborhood of $X \cap Z_0 \bmod Y \cap Z_0$ in Z_0. If either, (a) every triangulation of Z_0 extends to a triangulation of Z, or (b) $Z - (X \cup Y)$ is a manifold and $Z_0 - (X \cup Y)$ is an open subset of its boundary, then there is a polyhedron $V < Z$ such that (V, V_0) is a regular neighborhood of $X \bmod Y$ in (Z, Z_0).*

Proof. Let (W, W_0) be any regular neighborhood of $X \bmod Y$ in (Z, Z_0). By (7.7) or (7.8) there is a homeomorphism h of (Z, Z_0) onto itself such that $h | (X \cup Y) = 1$ and $h(W_0) = V_0$. Set $V = h(W)$,   q.e.d.

PROPOSITION 7.10. *Suppose that X, Y, $Z_0 < Z$. Then*

(a) *If (V, V_0) is a regular neighborhood of $X \bmod Y$ in (Z, Z_0) then V is a regular neighborhood of $(X \cup V_0) \bmod (Y \cup \dot{V}_0)$ in Z. (Here $\dot{V}_0 \equiv \mathrm{Bdy}_{Z_0} V_0$.)*

(b) *If V_0 is a regular neighborhood of $(X \cap Z_0) \bmod (Y \cap Z_0)$ in Z_0 which extends to some regular neighborhood (W, V_0) of $X \bmod Y$ in (Z, Z_0), if $(V \cap Z_0) - Y = V_0 - Y$, and if V is a regular neighborhood of $(X \cup V_0) \bmod (Y \cup \dot{V}_0)$ in Z then (V, V_0) is a regular neighborhood of $X \bmod Y$ in (Z, Z_0).*

REMARK. The hypothesis of (b) can sometimes be verified using (7.9).

Proof of (a). Let $X_1 = (X \cup V_0)$ and $Y_1 = (Y \cup \dot{V}_0)$. Then V is a topological neighborhood of $(X_1 - Y_1)$ and $(V \cap Y_1) = (Y_1)_R$. Choose a full triangulation $(J, J_0, K, L; h)$ of (Z, Z_0, X, Y) such that $(V, V_0) = h(N(K-L, J'), N(K-L, J_0'))$. Set $K_1 = K' \cup N(K-L, J_0')$ and $L_1 = L' \cup \dot{N}(K-L, J_0')$. Then (except for a technicality which we explain below), $(J', K_1, L_1; h)$ is a full triangulation of (Z, X_1, Y_1) such that

$$V = hN(K-L, J') = hN(K_1 - L_1, J') \quad \text{and} \quad \dot{N}(K-L, J') = \dot{N}(K_1 - L_1, J').$$

The last equation can be used to prove that $\dot{N}(K_1 - L_1, J') - L_1$ is locally collared in $N(K_1 - L_1, J')$ and in $C(K_1 - L_1, J')$. Therefore we may apply the stellar neighborhood theorem (6.1c) and conclude that V is a regular neighborhood of X_1 mod Y_1 in Z.

The assertions made in the preceding paragraph can be verified in a straightforward manner, except for the claim that L_1 is full in J'. This is not true in general but is true if

$$(*): L \cap N(K - L, J_0) < (J_0 \cap L)_R \equiv [\mathrm{Cl}\,(J_0 \cap K - J_0 \cap L)] \cap L.$$

Suppose that $(\bar{J}, \bar{J}_0, \bar{K}, \bar{L}; \bar{g})$ is any full triangulation of (Z, Z_0, X, Y). Let g be a homeomorphism of (Z, Z_0) fixed on $X \cup Y$ and carrying

$$(\bar{g}N(K - L, J'), \bar{g}N(K - L, J'_0))$$

onto (V, V_0). Then, by choosing $(J, J_0, K, L; h) = (\bar{J}', \bar{J}'_0, \bar{K}', \bar{L}'; g\bar{g})$ at the outset, we can insure that (*) is satisfied.

Proof of (b). Part (a) implies that W is a regular neighborhood of $(X \cup V_0)$ mod $(Y \cup \dot{V}_0)$ in Z. By hypothesis V is also such a regular neighborhood. In fact, by (7.1), (2.13), and the assumption that $[(V \cap Z_0) - Y] = [V_0 - Y]$, these are regular neighborhoods of $(X \cup V_0)$ mod $(Y \cup (\mathrm{Cl}\,(Z_0 - V_0)))$ in Z. Thus there is a homeomorphism $h: Z \to Z$ such that $h|(Z_0 \cup X \cup Y) = 1$ and $h(W) = V$. Hence (V, V_0) is a regular neighborhood of X mod Y in (Z, Z_0),   q.e.d.

8. Collapsibility and collapsible retractions.

DEFINITION. Suppose that $Y < X < Z$. Then we write $Z \searrow X$ (mod Y) if there exists a finite sequence $Z = Z_0 > Z_1 > Z_2 > \cdots > Z_q = X$ such that

(i) $Z_i = Z_{i+1} \cup B_i$ where B_i is an (n_i)-ball, $n_i > 0$.

(ii) $F_i = (B_i \cap Z_{i+1})$ is an $(n_i - 1)$-face of B_i.

(iii) $(F_i \cap Y) < \partial F_i$.

DEFINITION. If $Y < X < Z$ then $p: Z \to X$ is a *collapsible retraction*, mod Y, of Z onto X if

(i) p is a retraction of Z onto X.

(ii) $p^{-1}(x)$ is a compact collapsible polyhedron, for each x in X.

(iii) $p^{-1}(Y) = Y$.

The purpose of this section is to prove

THEOREM 8.1. *If $Y < X < Z$ is a triple of compact polyhedra then $Z \searrow X$ (mod Y) if and only if there is a collapsible retraction mod Y of Z onto X.*

In the light of (5.1) this theorem implies

COROLLARY 8.2. *If V is compact and V is a regular neighborhood of X mod Y in Z then $V \searrow X_R$ (mod Y_R).*

REMARK. At the end of this section we discuss the appropriate generalization of (8.1) to noncompact triples. The importance of collapsing relative to Y_R is indicated by Theorem 9.1 and the example given in [T].

LEMMA 8.3. *If $Z \searrow X$ and D^n is an n-ball then*

$$(Z \times D^n) \searrow (Z \times \partial D^n) \cup (X \times D^n).$$

Proof. Let $Z = Z_0 \searrow \cdots \searrow Z_q = X$ be a sequence of elementary collapses. Suppose that

$$(Z \times D^n) \searrow (Z \times \partial D^n) \cup (Z_i \times D^n) = [(Z \times \partial D^n) \cup (Z_{i+1} \times D^n)] \cup (B_i \times D^n)$$

where $Z_i = Z_{i+1} \cup B_i$, $(B_i \cap Z_{i+1}) = F_i$ and $G_i = \text{Cl}\,(\partial B_i - F_i)$. Notice that

$$[(Z \times \partial D^n) \cup (Z_{i+1} \times D^n)] \cap (B_i \times D^n) = (B_i \times \partial D^n) \cup (F_i \times D^n)$$
$$= \text{Cl}\,[\partial(B_i \times D^n) - (G_i \times D^n)], \text{ a face of } B_i \times D^n.$$

Thus $(Z \times D^n) \searrow (Z \times \partial D^n) \cup (Z_{i+1} \times D^n)$. The proof now proceeds by induction on the number of collapses,   q.e.d.

LEMMA 8.4. *If J and K are finite simplicial complexes such that $J \searrow K$ and if σ is a simplex, then $\sigma J \searrow \dot\sigma J \cup \sigma K$ (mod σ).*

The proof is similar to the proof of (8.3).

LEMMA 8.5. *If V is compact and is a regular neighborhood of X mod Y in Z then $V \searrow X_R$.*

Proof. Let $(J, K, L; h)$ be a full triangulation of (Z, X, Y) with $V = hN(K-L,J')$. Then $N(K-L, J') = N(K_R, N(K-L, J'))$. If A is a simplex of $\dot N(K_R, N(K-L, J'))$, then $A = b(\sigma_0) \cdots b(\sigma_r)$ where $\sigma_i = \alpha_i \beta_i \gamma_i$, as usual, and $\beta_0 \neq \varnothing \neq \gamma_0$. Hence (see 2.10)

$$\text{Lk}\,(A, N(K-L, J')) \cap K_R' = (\alpha_0 \beta_0)' \searrow 0.$$

By Theorem 2 of [W], $N(K-L, J') \searrow K_R'$,   q.e.d.

Proof of the sufficiency in Theorem 8.1. We assume that $Y < X < Z$ and that $p: Z \to X$ is a collapsible retraction mod Y. We proceed by induction on $q = $ dimension Y.

Suppose that Y is empty ($q = -1$). Choose a triangulation (J, K) of (Z, X) (we suppress the triangulating homeomorphism) such that $p: J \to K$ is simplicial. Choose first and second deriveds such that $p: J' \to K'$ and $p: J'' \to K''$ are also simplicial. Let σ be a principal simplex of K. We shall show that $p^{-1}(\sigma) \searrow \sigma \cup p^{-1}(\dot\sigma)$. Repeated application of this result then yields a collapse of J onto K.

Let $\sigma_0 = N(b(\sigma), \sigma'')$. Let C denote the collar $N(\dot\sigma'', \sigma'')$ and let $C \cap \sigma_0 = \dot\sigma_0$. Notice first that

$$(p^{-1}(\sigma_0), p^{-1}(\dot\sigma_0), \sigma_0) \simeq (\sigma_0 \times p^{-1}b(\sigma), \dot\sigma_0 \times p^{-1}b(\sigma), \sigma_0 \times b(\sigma)),$$

using the well-known fact that a simplicial mapping is a product on the inverse image of the interior of a top dimensional simplex of the range. Therefore we may apply (8.3) and conclude that $p^{-1}(\sigma_0) \searrow (p^{-1}(\dot\sigma_0) \cup \sigma_0)$. Now notice that

$(p^{-1}C, C) = (N(p^{-1}(\dot{\sigma})'', p^{-1}(\sigma)''), C)$ is a regular neighborhood of $p^{-1}(\dot{\sigma})$ in $(p^{-1}(\sigma), \sigma)$. Hence, by (7.10), $p^{-1}(C)$ is a regular neighborhood of

$$(p^{-1}(\dot{\sigma}) \cup C) \bmod \dot{\sigma}_0 \quad \text{in } p^{-1}(\sigma).$$

Therefore, by (8.5), $p^{-1}(C) \searrow p^{-1}(\dot{\sigma}) \cup C$. Putting these steps together we have

$$p^{-1}(\sigma) = p^{-1}(\sigma_0) \cup p^{-1}(C), \qquad [p^{-1}(\sigma_0) \cap p^{-1}(C) = p^{-1}(\dot{\sigma}_0)]$$
$$\searrow \sigma_0 \cup p^{-1}(C), \qquad [\sigma_0 \cap p^{-1}(C) = \dot{\sigma}_0]$$
$$\searrow \sigma_0 \cup (C \cup p^{-1}\dot{\sigma})$$
$$= \sigma \cup p^{-1}\dot{\sigma}.$$

Suppose now that $q \geq 0$ and the theorem is known for integers less than q. Let (J, K, L) be a triangulation of (Z, X, Y) such that $p: J \to K$ and $p: J' \to K'$ are simplicial. The hypotheses that $p^{-1}L = L$ implies (2.15) that $p^{-1}\dot{N}(L', K') = \dot{N}(L', J')$ and that $p^{-1}\text{Lk}\,(A, K') = \text{Lk}\,(A, J')$, for each simplex $A < L'$. Denote $I(A, S) = \text{Lk}\,(A, S') \cap \dot{N}(L', S')$. Then $p|I(A, J)$ is a collapsible retraction onto $I(A, K)$. So, using the result for $q = -1$, $I(A, J) \searrow I(A, K)$. Therefore (8.4) implies that $AI(A, J) \searrow AI(A, K) \cup \dot{A}I(A, J)$, (mod A). Moreover, if A and B are q-simplexes of L' then $[AI(A, J) \cap BI(B, J)] < [\dot{A}I(A, J) \cap \dot{B}I(B, J)]$ so that the collapses of $AI(A, J)$ and $BI(B, J)$ do not affect one another. Performing this collapse for each q-simplex $A < L'$ we have

$$J' = C(L', J') \cup N(L', J') = C(L', J') \cup \bigcup \{AI(A, J) \mid A < L'\}$$
$$\searrow C(L', J') \cup K' \cup \bigcup \{AI(A, J) \mid A < L', \dim A < q\} \,(\bmod\, L')$$
$$= K' \cup p^{-1}[C(L', K') \cup \bigcup \{AI(A, K) \mid A < L', \dim A < q\}]$$
$$\searrow K' \,(\bmod\, L'), \text{ by induction hypothesis.}$$

This completes the proof of the sufficiency in Theorem 8.1.

LEMMA 8.6. *The composition of collapsible retractions is a collapsible retraction.*

Proof. Suppose that $p_1: X \to X_1$ and $p_2: X_1 \to X_2$ are collapsible retractions and that $x \in X_2$. Then

$$(p_2 p_1)^{-1}(x) = p_1^{-1}(p_2^{-1}x) \searrow p_2^{-1}(x) \searrow x.$$

The first collapse comes from the assumption on p_1 and the sufficiency in (8.1). The second collapse comes from the assumption on p_2,   q.e.d.

Proof of the necessity in (8.1). Assume that $Z \searrow X \,(\bmod\, Y)$. Thus $Z = Z_0 > \cdots > Z_q = X$ where $Z_i = Z_{i+1} \cup B_i$ and $B_i \cap Z_{i+1} = F_i$. Define the retraction $p_i: Z_i \to Z_{i+1}$ as follows. Triangulate ∂F_i. Triangulate F_i as $v_0(\partial F_i)$ and triangulate B_i as $v_0 v_1(\partial F_i)$. Then p_i is defined to be the identity on Z_{i+1} and p_i is defined on B_i as the simplicial mapping such that $p_i(v_0 v_1) = v_0$. Notice that $p_i^{-1}(\partial F_i) = \partial F_i$ and $p_i^{-1}(x)$ is a point or an arc for each $x \in Z_{i+1}$. Then $p = p_{q-1} \cdots p_0$ is a retraction of Z onto X. Point inverses are collapsible by (8.6). The fact that $(Y \cap F_i) < \partial F_i$ implies that no p_i ever brings a point outside of Y into Y. Hence $p^{-1}(Y) = Y$,   q.e.d.

REMARK. Theorem 8.1 asserts the equivalence, in the compact case, of the existence of a collapse $X \searrow Y \pmod Z$ and the existence of a collapsible retraction, mod Y, of Z onto X. It is the latter notion which we choose to emphasize in the general case. Notice however that the proof of (8.1) gives us the following information: If a given infinite collapsing process[5] induces a well-defined retraction p such that, for each x, $p|p^{-1}(x)$ is the composition of finitely many collapsible retractions then p is a collapsible retraction. On the other hand, if there is a collapsible retraction mod Y^q of Z onto X^n then there is a composition of $n+q+2$ "collapses" taking Z onto X, where a "collapse" consists of countably many disjoint, simultaneous finite collapses relative to Y.

9. Regular neighborhoods via collapsibility.

The purpose of this section is to prove

THEOREM 9.1. *If X, Y, $V < M$ and*

(1) *$M - (X \cup Y)$ is an n-manifold,*

(2) *V is a topological neighborhood of $X - Y$ in M,*

(3) *$V \cap Y = Y_R$,*

(4) *$V - X$ is an n-manifold,*

(5) *there is a collapsible retraction, mod Y_R, of V onto X_R, then*

(a) *there are regular neighborhoods U and U_0 of X mod Y in M, contained in V, and there is an isotopy H of V into itself such that $H|(V \times 0) \cup (U_0 \times I) = 1$ and $H_1(V) = U$,*

(b) *if $M - (X \cup Y)$ is a manifold without boundary then V is a regular neighborhood of X mod Y in M,*

(c) *if M is actually a manifold and if $\mathrm{Cl}\,[(V \cap \partial M) - Y]$ is a regular neighborhood of $(X \cap \partial M)$ mod $(Y \cap \partial M)$ in ∂M then V is a regular neighborhood of X mod Y in M.*

REMARK. One can combine (9.1), (7.10) and (7.9) to get the following program for recognizing regular neighborhoods *in polyhedra* via collapsibility. If Z is an n-dimensional polyhedron let Z_1 denote the set of points of Z whose links are not $(n-1)$-spheres (i.e. the intrinsic $(n-1)$-skeleton of Z). If X, Y, $V < Z$ let $V_1 = \mathrm{Cl}\,[(V \cap Z_1) - Y]$. Then, by (7.10b), V is a regular neighborhood of X mod Y in Z if (a) V_1 is a regular neighborhood of $(X \cap Z_1)$ mod $(Y \cap Z_1)$ in Z_1, (b) V_1 extends to some regular neighborhood of X mod Y in (Z, Z_1), and (c) V is a regular neighborhood of $(X \cup V_1)$ mod $Y \cup \mathrm{Cl}\,(Z_1 - V_1)$ in Z. Since $(Z - Z_1)$ is an n-manifold without boundary, condition (c) can be checked using (9.1b). Condition (b) is not always satisfied. One has to verify it directly or, if possible, apply (7.9). Condition (a) is just the recognition problem in lower dimension. Proceed inductively[6].

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(5) E.g. the process of trace-finite collapse considered in [S].

(6) Since this was written J. Stallings [*Notes on polyhedral topology*, Tata Institute, 1967] and E. Akin (loc. cit.) have carried through similar but more efficient programs for compact polyhedra.

EXAMPLE.   The importance of condition (b) in the above remark is illustrated in the simple situation where $Z=(R^2 \times 0) \cup ([0, 2] \times 0 \times [0, 10]) < R^3$, $X=\{(1, 0, 0)\}$, $Y= \varnothing$, and $V=([0, 2] \times 0 \times [0, 5]) \cup ([-1, 3] \times [-1, 1] \times 0)$. An example where (c) fails was given in (5.5).

CONVENTION.   If Z is an arbitrary q-dimensional polyhedron then ∂Z denotes the subpolyhedron consisting of points of Z whose links are not $(q-1)$-spheres (i.e. the intrinsic $(q-1)$-skeleton of Z. Notice that ∂Z underlies a subcomplex of any triangulation of Z.

Preliminaries to proving (9.1a).   Let $(J, K, L; h)$ be a triangulation of (V, X_R, Y_R) such that $h^{-1}ph: J \to K$ is simplicial. To simplify the notation we write $p=h^{-1}ph$, when no confusion can occur. Choose a first derived J' such that $p: J' \to K'$ is also simplicial. The proofs given to derive the basic facts about dual cells when J is a manifold (see §5 of [C]) apply almost verbatim here, where $J-K$ is a manifold and the symbol "∂" has the broadened meaning given above. Thus, if σ is an i-simplex of $K-L$, we know

(A) $D(\sigma, p) - D(\sigma, K)$ is an $(n-i)$-manifold (possibly empty).

(B) $\partial D(\sigma, p) = \dot{D}(\sigma, p) \cup D(\sigma, p|\partial J)$.

(C) $\dot{D}(\sigma, p) - D(\sigma, K)$ and $D(\sigma, p|\partial J) - D(\sigma, K)$ are $(n-i-1)$-manifolds.
      (One or both may be empty.)

Analogous statements hold for $D(\sigma, J)$, $\dot{D}(\sigma, J)$ and $D(\sigma, \partial J)$.

The idea of the proof of (a) is to retriangulate $|J|$ as a stellar neighborhood of K' mod L'. First we use the fact that p is a collapsible retraction to triangulate each dual cell $D(\sigma, p)$, $\sigma < K-L$, as a cone which stretches from the subcone $D(\sigma, K)$ out to ∂J. This yields a new triangulation of $|C(L', J')|$. Then we use the hypothesis that $p^{-1}L=L$ to extend this triangulation over $|N(L', J')|$. The crux of the first part is given by

LEMMA 9.2.   *If, in the above situation, $\sigma < K-L$ then there is a homeomorphism*

$$f: (vC(b(\sigma), \partial D(\sigma, p)), v\dot{D}(\sigma, K)) \to (D(\sigma, p), D(\sigma, K))$$

such that

(i) $f|C(b(\sigma), \partial D(\sigma, p))=1$,

(ii) $f(v)=b(\sigma)$ *and* $f|v\dot{D}(\sigma, K)=(f|v) * 1_{\dot{D}(\sigma,K)}$ *(see Figure 3)*.

REMARK. $b(\sigma) < \partial D(\sigma, p) \Leftrightarrow \sigma < \partial J$. If $\sigma \nless \partial J$ then

$$C(b(\sigma), \partial D(\sigma, p)) = \partial D(\sigma, p).$$

Similar assertions hold for $\partial D(\sigma, J)$.

Proof of the lemma.   Since $p|D(\sigma, p)$ is a collapsible retraction of $D(\sigma, p)$ onto $D(\sigma, K)$ it follows (8.1) that $D(\sigma, p) \searrow D(\sigma, K)$. We subdivide to get a simplicial collapse and then apply Whitehead's shelling procedure [W, Theorem 23n] to conclude that $D(\sigma, p)$ is homeomorphic, keeping $D(\sigma, K)$ fixed, to a regular neighborhood of $D(\sigma, K)$ in $D(\sigma, p)$. (Whitehead's procedure works because
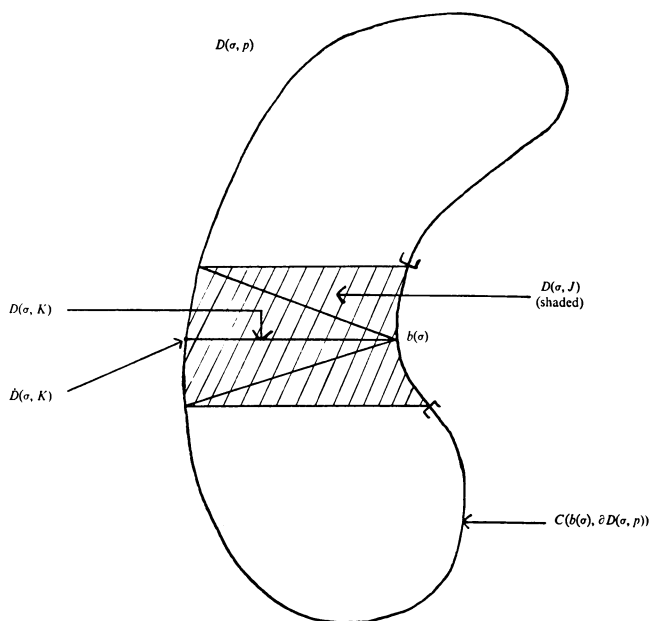
A schematic diagram of $D(\sigma, p)$

FIGURE 3

$[D(\sigma, p) - D(\sigma, K)]$ is a manifold. This procedure only involves the subtracting away of balls which are stars of barycenters of simplexes *deleted* during the simplicial collapse. Hence it does not matter whether $D(\sigma, p)$ is a manifold along $D(\sigma, K)$.)

We claim that $D(\sigma, J)$ is also homeomorphic, keeping $D(\sigma, K)$ fixed, to a regular neighborhood of $D(\sigma, K)$ in $D(\sigma, p)$. For one checks easily that

$$D(\sigma, J) = N(D(\sigma, K), D(\sigma, p)) = N(D(\sigma, K), D(\sigma, J))$$

and

$$\dot{N}(D(\sigma, K), D(\sigma, p)) = \dot{N}(D(\sigma, K), D(\sigma, J))$$
$$< [\dot{D}(\sigma, J) - D(\sigma, K)]$$
$$< [\partial D(\sigma, J) - D(\sigma, K)].$$

But $\dot{N}(D(\sigma, K), D(\sigma, J))$ is a manifold by (5.4), being simplicially isomorphic to $C(\mathrm{Lk}\,(\sigma, K)', \mathrm{Lk}\,(\sigma, J)')$. Thus it is a codimension zero submanifold of $[\partial D(\sigma, J) - D(\sigma, K)]$ and so is locally collared in $[D(\sigma, J) - D(\sigma, K)]$. The claim follows from the stellar neighborhood theorem (6.1a).

Invoking the uniqueness of regular neighborhoods, there is a homeomorphism $f_0 \colon D(\sigma, p) \to D(\sigma, J)$ such that $f_0 | D(\sigma, K) = 1$. We shall modify f_0 to get a homeomorphism which also takes $C(b(\sigma), \partial D(\sigma, p))$ onto $C(b(\sigma), \partial D(\sigma, J))$.

Notice that

$$D(\sigma, K) \cap \partial D(\sigma, p) = D(\sigma, K) \cap [\dot{D}(\sigma, p) \cup D(\sigma, p | \partial J)]$$
$$= D(\sigma, K) \cap [p^{-1}\dot{D}(\sigma, K) \cup (p^{-1}D(\sigma, K) \cap \partial J)]$$
$$= \dot{D}(\sigma, K) \cup (D(\sigma, K) \cap \partial J)$$
$$= \dot{D}(\sigma, K) \cup D(\sigma, K \cap \partial J).$$

Similarly

$$D(\sigma, K) \cap \partial D(\sigma, J) = \dot{D}(\sigma, K) \cup D(\sigma, K \cap \partial J).$$

We claim that $N(b(\sigma), \partial D(\sigma, p)) = N_\sigma$ is a regular neighborhood of

$$D(\sigma, K \cap \partial J) \bmod \dot{D}(\sigma, K)$$

in $\partial D(\sigma, p)$. For the complex $D(\sigma, K \cap \partial J)$ is a subcone of N_σ and it is easy to see that

$$D(\sigma, K \cap \partial J) \cap \dot{D}(\sigma, K) = D(\sigma, K \cap \partial J) \cap \dot{N}_\sigma,$$
$$N_\sigma = N(D(\sigma, K \cap \partial J) - \dot{D}(\sigma, K), \partial D(\sigma, p)).$$

Further, $[\dot{N}_\sigma - \dot{D}(\sigma, K)]$ is locally collared in the cone N_σ and so, as a codimension one submanifold of the open manifold $[\partial D(\sigma, p) - D(\sigma, K)]$, is also locally collared in $C(b(\sigma), \partial D(\sigma, p))$. Thus the claim follows from the stellar neighborhood theorem (6.1c). Similarly $N(b(\sigma), \partial D(\sigma, J))$ is a regular neighborhood of

$$D(\sigma, K \cap \partial J) \bmod \dot{D}(\sigma, K) \quad \text{in } \partial D(\sigma, J).$$

But $f_0 | D(\sigma, K) = 1$. Thus the preceding paragraph tells us that $f_0 N(b(\sigma), \partial D(\sigma, p))$ and $N(b(\sigma), \partial D(\sigma, J))$ are each regular neighborhoods of $D(\sigma, K \cap \partial J) \bmod \dot{D}(\sigma, K)$ in $\partial D(\sigma, J)$. Since $D(\sigma, K \cap \partial J) \cup \dot{D}(\sigma, K) = D(\sigma, K) \cap \partial D(\sigma, J)$ it follows from (7.8) that there is a homeomorphism f_1 of $D(\sigma, J)$ onto itself such that

$$f_1 | D(\sigma, K) = 1$$

and

$$f_1 f_0 N(b(\sigma), \partial D(\sigma, p)) = N(b(\sigma), \partial D(\sigma, J)).$$

The homeomorphism $f_2 = f_1 f_0$ of $D(\sigma, p)$ onto $D(\sigma, J)$ takes $C(b(\sigma), \partial D(\sigma, p))$ onto $C(b(\sigma), \partial D(\sigma, J))$ and is the identity on $D(\sigma, K)$.

Since $D(\sigma, J) = b(\sigma)\dot{D}(\sigma, J) = b(\sigma)C(b(\sigma), \partial D(\sigma, J))$ is a cone with $D(\sigma, K)$ as a subcone, $D(\sigma, p)$ is, piecewise linearly, also such a cone. Explicitly, define

$$f: (vC(b(\sigma), \partial D(\sigma, p)), v\dot{D}(\sigma, K)) \to (D(\sigma, p), D(\sigma, K))$$

by setting $f(v) = b(\sigma)$ and defining

$$f = f_2^{-1} \circ [(f | v) * (f_2 | C(b(\sigma), \partial D(\sigma, p)))].$$

This completes the proof of the lemma.

Proof of (9.1a). Let $\bar{L}$ and Q be joinable complexes and let $\bar{C}$ be joinable with $\bar{L} * Q$, where $\bar{L} \cong L'$, $Q \cong C(L', K')$ and $\bar{C} \cong C(K', \partial J')$. Every simplex of K' is of the

form AB where $A < L'$ and $B < C(L', K')$, so K' is naturally isomorphic to a sub-complex $\bar{K} < \bar{L} * Q$. Let $f: \bar{K} \cup \bar{C} \to K' \cup C(K', \partial J')$ be a simplicial isomorphism such that $f(\bar{K}) = K'$ and $f(\bar{L}) = L'$. In general, if $S < K'$ we set $\bar{S} = f^{-1}(S)$. We shall construct a subcomplex $\bar{J} < \bar{K} * \bar{C}$ and a homeomorphism $g: \bar{J} \to J'$ such that $(\bar{J}, \bar{K}, \bar{L}; hg)$ is a stellar triangulation of (V, X_R, Y_R) and $hg(\dot{N}(\bar{K} - \bar{L}, \bar{J}) - \bar{L})$ is locally collared in V. Then (9.1a) will follow from the stellar neighborhood theorem.

Let $J_i < J'$ $(-1 \le i \le q = \dim(K - L))$ be given by

$$J_i = K' \cup C(K', \partial J') \cup \bigcup \{D(\sigma^j, p) | \sigma^j < K - L, j \ge q - i\}.$$

Notice that $J_q = K' \cup C(L', J')$ because

$$
\begin{aligned}
C(L', J') &= p^{-1} C(L', K'), \quad \text{by (2.14)} \\
&= p^{-1} \cup \{D(\sigma, K) | \sigma < K - L\} \\
&= \bigcup \{D(\sigma, p) | \sigma < K - L\}.
\end{aligned}
$$

Then $(\bar{J}_{-1}, g_{-1}) \equiv (\bar{K} \cup \bar{C}, f)$ is a triangulation of J_{-1}. Proceeding inductively assume that a complex $\bar{J}_j < \bar{K} * \bar{C}$ and a homeomorphism g_j have been chosen for each $j < i$ $(i \le q)$ so that

    (i) $(\bar{J}_j, g_j)$ is a triangulation of J_j.

    (ii) If $k < j$ then $\bar{J}_k < \bar{J}_j$ and $g_k = g_j | \bar{J}_k$.

    (iii) If $\sigma = \sigma^{q-j} < K - L$ then $g_j^{-1} D(\sigma, p)$ is a subcomplex of $\bar{J}_j$—denoted $\bar{D}(\sigma, p)$—such that every principal simplex of $\bar{D}(\sigma, p)$ is the join of a nonempty simplex in $\bar{D}(\sigma, K)$ and a simplex in $\bar{C}$.

In order to extend $(\bar{J}_{i-1}, g_{i-1})$ to a triangulation which covers each $D(\sigma^{q-i}, p)$, notice (writing $\sigma = \sigma^{q-i}$) that

$$
\begin{aligned}
D(\sigma, p) \cap J_{i-1} &= D(\sigma, K) \cup [D(\sigma, p) \cap C(K', \partial J')] \cup \dot{D}(\sigma, p) \\
&= D(\sigma, K) \cup [D(\sigma, p) \cap C(K', \partial J')] \cup [\dot{D}(\sigma, p) \cup \dot{D}(\sigma, p | \partial J)] \\
&= D(\sigma, K) \cup \dot{D}(\sigma, p) \cup \dot{D}(\sigma, p | \partial J) \\
&\qquad\qquad \cup \{b(\tau) \cdots b(\psi) | \sigma < p(\tau), \tau \nless K, \psi < \partial J\} \\
&= D(\sigma, K) \cup \dot{D}(\sigma, p) \cup C(b(\sigma), D(\sigma, p | \partial J)) \\
&= D(\sigma, K) \cup C(b(\sigma), \dot{D}(\sigma, p) \cup D(\sigma, p | \partial J)) \\
&= D(\sigma, K) \cup C(b(\sigma), \partial D(\sigma, p)).
\end{aligned}
$$

Because $C(b(\sigma), \partial D(\sigma, p))$ is the union of the last two summands of the first equation, we see from (i)–(iii) that its inverse image underlies a subcomplex—call it T—of J_{i-1}. Also from the first equation, $T \cap \bar{K} = g_{i-1}^{-1} \dot{D}(\sigma, K)$, so that $\bar{b}(\sigma)T < \bar{D}(\sigma, K) * (T \cap \bar{C})$ is a subcomplex of $\bar{K} * \bar{C}$. We claim that $(\bar{b}(\sigma)T) \cap \bar{J}_{i-1} = \bar{D}(\sigma, K) \cup T$. This is because any simplex of $\bar{J}_{i-1}$ which meets both $\bar{C}$ and $\bar{K}$ lies in some $\bar{D}(\tau, p)$, where $\dim \tau > i$, and thus, by (iii), cannot contain $\bar{b}(\sigma)$. We define $\bar{D}(\sigma, p) = \bar{b}(\sigma)T$. We extend g_{i-1} to $g_i: \bar{D}(\sigma, p) \to D(\sigma, p)$ by stipulating that

$$g_i | \bar{b}(\sigma)T = f \circ (k * (g_{i-1} | T)),$$

where $k(\bar{b}(\sigma)) = v$ and $f: vC(b(\sigma), \partial D(\sigma, p)) \to D(\sigma, p)$ is the homeomorphism given by the preceding lemma. Doing this for all $\sigma = \sigma^{q-i} < K - L$ we get a triangulation $(\bar{J}_i, g_i)$ of J_i satisfying (i)–(iii).

The pair $(\bar{J}_q, g_q)$ constructed by this process is a new triangulation of $|K' \cup C(L', J')|$. We proceed to extend this over $|N(L', J')|$. Let us denote

$$S = \{(A, \sigma) \mid \varnothing \neq A = b(\alpha_0) \cdots b(\alpha_q) < L', \alpha_q < \sigma, \sigma < K - L\}.$$

Then

$$N(L', K') = \bigcup \{AD(\sigma, K) \mid (A, \sigma) \in S\}$$
$$N(L', J') = p^{-1}N(L', K'), \quad \text{by (2.14)}$$
$$= \bigcup \{AD(\sigma, p) \mid (A, \sigma) \in S\}.$$

Define $\bar{J} < \bar{K} * \bar{C}$ by

$$\bar{J} = \bigcup \{\bar{A}\bar{D}(\sigma, p) \mid (A, \sigma) \in S\} \cup \bar{J}_q.$$

Since $\bar{A}\bar{D}(\sigma, p) < \bar{A} * \bar{D}(\sigma, K) * (\bar{D}(\sigma, p) \cap \bar{C})$, $\bar{J}$ is a subcomplex of $\bar{K} * \bar{C}$. If $(A, \sigma) \in S$ and $\tau < K - L$ then $(\bar{A}\bar{D}(\sigma, p)) \cap \bar{D}(\tau, p) = [\bar{D}(\sigma, p) \cap \bar{D}(\tau, p)] = \bar{D}(\sigma \cdot \tau, p)$ where $\sigma \cdot \tau$ is the smallest simplex of K containing both σ and τ and $\bar{D}(\sigma \cdot \tau) \equiv \varnothing$ if there is no such simplex. Thus

$$N(\bar{L}, \bar{J}) = \bigcup \{\bar{A}\bar{D}(\sigma, p) \mid (A, \sigma) \in S\},$$
$$\dot{N}(\bar{L}, \bar{J}) = \bigcup \{\bar{D}(\sigma, p) \mid \exists A \ni (A, \sigma) \in S\},$$
$$N(\bar{L}, \bar{J}) \cap \bar{J}_q = \dot{N}(\bar{L}, \bar{J}) \cup N(\bar{L}, \bar{K}).$$

Define $g: \bar{J} \to J'$ by the conditions that

$$g|\bar{J}_q = g_q, \quad g|\bar{A}\bar{D}(\sigma, p) = (g_q|\bar{A}) * (g_q|\bar{D}(\sigma, p)) \quad \text{if } (A, \sigma) \in S.$$

Then g is a well-defined homeomorphism and $(\bar{J}, \bar{K}, \bar{L}; g)$ is a triangulation of $(|J'|, |K'|, |L'|)$.

Clearly $\bar{L} \lhd \bar{K} \lhd \bar{J}$. Also $\bar{J} = N(\bar{K} - \bar{L}, \bar{J})$ because every principal simplex of $\bar{J}$ is of the form $\bar{A}B$ where $\bar{A} < \bar{L}$ ($\bar{A}$ may be empty) and B is a principal simplex of $\bar{D}(\sigma, p)$ for some $\sigma < K - L$. Thus by (iii), $B \cap (\bar{K} - \bar{L}) \neq \varnothing$.

Finally $\dot{N}(\bar{K} - \bar{L}, \bar{J}) - \bar{L}$ is locally collared in $\bar{J}$. For, we claim that $g(\dot{N}(\bar{K} - \bar{L}, \bar{J}) - \bar{L}) = C(K' - L', \partial J') - L'$. The latter complex is locally collared in the manifold $J' - (K' \cup L')$ since, by (5.4) it is an $(n-1)$-dimensional submanifold of $\partial(J' - (K' \cup L'))$. To verify the claim, suppose that F is a simplex of $\dot{N}(\bar{K} - \bar{L}, \bar{J}) - \bar{L}$. Then $F = \bar{A}B$ where $\bar{A} < \bar{L}$ (possibly $\bar{A}$ is empty) and $B < \bar{D}(\sigma, p)$ for some $\sigma < K - L$. Actually $B < \bar{D}(\sigma, p) \cap \bar{C}$ since $F \cap (\bar{K} - \bar{L}) = \varnothing$. But $g|\bar{C}$ is a simplicial isomorphism. Hence $g(B)$ is a simplex of $C(K', \partial J') \cap D(\sigma, p)$ and $g(F) = A * g(B)$ is a simplex of J'. In fact $g(F) < \partial J'$ because $A < \partial J', g(B) < \partial J'$ and $\partial J'$ is full in J'. [Proof that $A < \partial J'$: Let $\tilde{L} = \text{Lk}(A, J')$. Then $\varnothing \neq g(B) < \tilde{L} \cap C(K', \partial J')$. Thus $p|\tilde{L} \neq 1$. But $p|\tilde{L}$ is a retraction and $\tilde{L} \searrow p\tilde{L}$. So $\tilde{L}$ is not a sphere.] Therefore $g(F) < \partial J', g(F) \not< L'$ since $F \not< \bar{L}$, and $g(F) \cap (K' - L') = g(F \cap (\bar{K} - \bar{L})) = \varnothing$. This proves that $g(F) < C(K' - L', \partial J') - L'$. The opposite inclusion is proven in a similar manner.

We apply the stellar neighborhood theorem to the triangulation $(\bar{J}, \bar{K}, \bar{L}; hg)$ of (V, X_R, Y_R). The proof of part (a) is complete.

Proof of (9.1b). We assume that $M-(X \cup Y)$ is a manifold without boundary. Hence, by invariance of domain and assumptions (2)–(4),

$$\partial(V-X) = \mathrm{Bdy}_{M-(X\cup Y)}(V-X) = \mathrm{Bdy}_{M-X}(V-X) = (\mathrm{Bdy}_M V)-Y.$$

Therefore $(\mathrm{Bdy}_M V)-Y$ is locally bicollared in M and

$$\partial(V-X) \cup Y_R = (\mathrm{Bdy}_M V) \cup Y_R.$$

Let W be a regular neighborhood of X mod Y in M. Arguing as above,

$$\partial(W-X) \cup Y_R = (\mathrm{Bdy}_M W) \cup Y_R.$$

By (9.1a) there exists a homeomorphism $f: V \to W$ such that $f|X_R=1$. Clearly $f(\partial(V-X))=\partial(W-X)$. If $(J, K, L; h)$ is a stellar triangulation of (W, X_R, Y_R) in which $h\dot{N}(K-L, J)=(\mathrm{Bdy}_M W) \cup Y_R$ then $(J, K, L; f^{-1}h)$ is an analogous triangulation of (V, X_R, Y_R). The result now follows from the stellar neighborhood theorem (6.1b).

Proof of (9.1c). We now assume hypotheses (2)–(5) of (9.1) and also that M is a manifold and that $V_1 \equiv \mathrm{Cl}\,[(V \cap \partial M)-Y]$ is a regular neighborhood of

$$(X \cap \partial M) \bmod (Y \cap \partial M)$$

in ∂M. Let W be a regular neighborhood of X mod Y in M. We denote

$$X_1 = \mathrm{Cl}\,[(X \cap \partial M)-(Y \cap \partial M)] = (X \cap \partial M)_R,$$
$$Y_1\dot{} = X_1 \cap (Y \cap \partial M) = (Y \cap \partial M)_R,$$
$$W_1 = \mathrm{Cl}\,[(W \cap \partial M)-Y].$$

By part (a) there is a homeomorphism $f: V \to W$ such that $f|X_R=1$. Clearly $f(\partial V)=\partial W$ (where, as before, ∂V and ∂W denote the intrinsic $(n-1)$-skeletons of V and W). We shall modify f to get a new homeomorphism which also takes V_1 onto W_1.

We claim that V_1 is a regular neighborhood of X_1 mod $(Y \cap \partial V)$ in ∂V. For, since V_1 is a regular neighborhood of $(X \cap \partial M)$ mod $(Y \cap \partial M)$ in ∂M, there is a stellar triangulation $(J, K, L; h)$ of

$$(V_1, X_1, Y_1) = (V_1, \mathrm{Cl}\,(X_1-(Y \cap \partial V)), \mathrm{Cl}\,(X_1-(Y \cap \partial V)) \cap (Y \cap \partial V))$$

such that $h\dot{N}(K-L, J)=(\mathrm{Bdy}_{\partial M} V_1) \cup Y_1=(\mathrm{Bdy}_{\partial V} V_1) \cup Y_1$. [The last equality follows from invariance of domain since it is equivalent to the assertion that

$$(\mathrm{Int}_{\partial M} V_1)-Y_1 = (\mathrm{Int}_{\partial V} V_1)-Y_1$$

where ∂M and $(\partial V)-Y$ are $(n-1)$-manifolds without boundary and $V_1 \cap Y=Y_1$.] Thus the claim follows from the stellar neighborhood theorem (6.1b) once we know that $(\mathrm{Bdy}_{\partial V} V_1)-(Y \cap \partial V)$ is locally bicollared in ∂V. This is true because

the latter set is just $\text{Bdy}_{(\partial V)-Y}(V_1 - Y)$ where $V_1 - Y$ is an $(n-1)$-manifold in the open $(n-1)$-manifold $(\partial V) - Y$.

The argument above also applies to W_1 (see (2.13)). Thus $f(V_1)$ and W_1 are both regular neighborhoods of $X_1 \bmod (Y \cap \partial W)$ in ∂W. But $(X_R \cap \partial W) = X_1 \cup (Y \cap \partial W)$ and $(W - X_R)$ is an n-manifold with $(\partial W - X_R)$ as an open subset of its boundary. Hence, by (7.8) there is a homeomorphism $G: W \to W$ such that $G|X_R = 1$ and $Gf(V_1) = W_1$. Let $h = Gf$.

Since $h(V_1) = W_1$ it follows that $h(V, V \cap \partial M) = (W, W \cap \partial M)$. Thus, by invariance of domain,

$$h((\text{Bdy}_M V) \cup Y_R) = (\text{Bdy}_M W) \cup Y_R.$$

Let $(J, K, L; h_0)$ be a stellar triangulation of (W, X_R, Y_R) such that $h_0 \dot{N}(K-L, J) = (\text{Bdy}_M W) \cup Y_R$. Then $(J, K, L; h^{-1}h_0)$ is such a triangulation of (V, X_R, Y_R). Since $(\text{Bdy}_M V) - Y = \text{Bdy}_{M-Y}(V - Y)$, and since $(V - Y)$ is an n-dimensional submanifold of $(M - Y)$ meeting $\partial(M - Y)$ in an $(n-1)$-dimensional submanifold, we know that $(\text{Bdy}_M V) - Y$ is locally bicollared in M. Hence, by the stellar neighborhood theorem, V is a regular neighborhood of $X \bmod Y$ in M,   q.e.d.

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